

# THE GROUP THEORETICAL DESCRIPTION OF THE THREE-BODY PROBLEM

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The group theoretical description of the three-particle problem provides successful techniques for the solution of different questions. We present here a review of this approach.

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## 1. Introduction

The three-body problem in quantum mechanics in general, and in topics like molecular, nuclear and particle physics in particular, provides a variety of interesting fields for investigations. The developed technics were useful for the solution of different, sometimes quite unexpected problems, and because of that the three-body problem was and remains a rather attractive topic.

In the thirties and forties the Hartree-Fock, Thomas-Fermi, and variational methods were suggested, and many different approaches were considered in the past decades.<sup>1-3</sup> Developments and improvements have never stopped; see, for example, Refs. 4, 5, 6.

In classical mechanics the three-body problem appeared long ago when the motion of planets became the subject of investigations. A solution for newtonian interactions was given by Euler<sup>7</sup> around 1760; a little later Lagrange<sup>8</sup> solved a generalized problem with additional linear forces. The story of further developments can be found, e.g. in Refs. 9, 10.

In quantum mechanics the three-particle problem was taken into consideration from the very beginning, in the end of the twenties. In the thirties the problem of falling on the center was formulated.<sup>11</sup> In the fifties and the sixties a set of specific topics were investigated. First, that was the formulation of the equation for calculations of energy levels in three-nucleon systems,  $\text{He}^3/\text{H}^3$ , using point-like forces for two-particle interactions, the Skornyakov – Ter-Martirosyan equation.<sup>12</sup>

Its analysis and the determination of the neutron-deuteron scattering length was given by Danilov.<sup>13</sup> The general solution of the equation was formulated by Minlos and Faddeev.<sup>14</sup> Namely, there are two sets of levels below and above the value of the basic one (which is not determined by the equation, it is a parameter of the model). The lower set corresponds to the three-body collapse,<sup>11</sup> while the second set corresponds to the concentration of levels towards zero binding energy. The effect of concentration of the three-body spectra at zero total energy with increasing two-particle scattering lengths was emphasized in.<sup>15</sup> The three-body collapse branch of levels was eliminated in the Efimov solution<sup>15</sup> by a cutting procedure at large relative momenta – this method of regularization implicitly introduces short range three-body forces.

At present the short-range approach (Skornyakov–Ter-Martirosyan equation) with a set of levels concentrating at zero energy is widely used in molecular physics, see e.g. Refs. 16, 17, 18, 19 and references therein.

The Faddeev equation<sup>20</sup> for non-relativistic three-particle systems and that for four particles<sup>21</sup> are logical consequences of the investigation of many particle systems in the framework of quantum mechanics.<sup>22</sup>

In the sixties the dispersion relation technics was used for the investigation of three-particle systems: first, within the non-relativistic approach, to expand the amplitudes near the thresholds,<sup>23,24</sup> then to describe certain relativistic processes with resonances in the intermediate state.<sup>25,26</sup>

The relativistic three-particle dispersion relation equation was written in Refs. 27, 28, see also Ref. 29. Relativistic three-body equations were seriously discussed before<sup>30</sup> but at that time the analytical continuation of the amplitude situated it on the second (*i.e.*, unphysical) sheets of complex variables.

As a rule, the relativistic description of a three particle system is also a description of the surrounding states which are related to each other by transitions. The problem of coupled states is discussed in Ref. 29, and this line of research requires significant efforts.

A further way for the development of methods for investigating three particle systems is the elaboration of techniques for the expansion of wave functions (or amplitudes) over a convenient set of states. It is a problem both for the non-relativistic and relativistic approaches. Different sets of expansion are appropriate for different problems depending on the type of interactions in the system.

In this review paper we present the group theoretical description of the three-particle problem suggested in the middle of the sixties;<sup>31–36</sup> the last papers on the subject were written in the eighties. It is just a reminder of this part of the three-body story. It can be considered as an addition to the book.<sup>29</sup>

From a group theoretical point of view the most interesting questions are related to the fifth quantum number  $\Omega$ . This has to be introduced because the quantum numbers describing rotations and permutations are not sufficient to characterize the states in the three-body system. In the considered papers a complete set of basis functions for the quantum mechanical three-body system is chosen in the form of

hyperspherical functions, characterized by quantum numbers corresponding to the chain  $O(6) \supset SU(3) \supset O(3)$ . Equations are derived to obtain the basis functions in an explicit form.

The problem of constructing a basis for a system of three free particles, making use of representations of the three-dimensional rotation group and of the permutation group, is quite simple in principle. Nevertheless, problems appear in implementing a straightforward way for the construction of a general solution for the set of equations which determine the eigenfunctions. As it turns out, the eigenvalue equations can be simplified considerably, then the solution is derived in a closed form, the coefficients are calculated in different ways, numerical results are obtained.

The developed technics can be applied in a variety of cases. First of all, as soon as the quantum mechanical problem which we have considered has the same symmetry properties as the classical one, it is possible to investigate the classical problem. The equations of motion are obtained very easily for both the case of free particles and that of different potentials.

The classification of a three-body system can be used also for the analysis of three-particle decay processes. For example, dealing with a Dalitz plot for decay processes, it turned out to be useful to expand the point density inside the physical region into a series of orthonormal functions. (Such an expansion is similar to the usual phase analysis for two-particle decays. It was helpful in analyzing experimental data, for the calculation of different correlation functions etc.) The set of basis functions chosen as K-harmonics was especially suitable for the description of correlations between the momenta of particles. Also, from a practical point of view it was essential to develop a method to calculate matrix elements of two-particle interactions introducing different potentials and to obtain a proper approximation for bound states as well.

Let us note that investigation of the question for a special case was carried out by Badalyan and Simonov.<sup>37,38</sup> The basis for expansion was formed by the so-called K-polynomials, which are harmonic functions corresponding to the Laplace operator on the six-dimensional sphere. Further, a complete set of solutions was considered with five commuting operators. It was shown how in principle one can construct the polynomials being eigenfunctions of these operators. But the calculations were done in a rather complicated way, so only the lowest polynomials were obtained, which can be characterized by four quantum numbers.

Another possibility of constructing a basis was demonstrated by Zickendraht,<sup>39</sup> however the used method is also too complicated and does not give a sufficiently general result.

In the paper of Lévy-Leblond and Lévy-Nahas<sup>40</sup> the connection between the basis and the representations of  $SU(3)$  is pointed out. The authors have used a proper parametrization and obtained the Laplace operator expressed in terms of angular variables. Yet, they did not discuss a general solution either.

If one intends to construct harmonic functions for the three-particle system analogous to the spherical functions forming the basis in case of two particles, it

is natural to use angular variables on the six-dimensional sphere or on the three-dimensional complex sphere and build up the required functions in terms of these coordinates. In this survey we demonstrate a way to carry out this program. Let us note that the full group of motion on the six-dimensional sphere is too large for our aim. The problem is just to find the suitable subgroup.

Introducing angular variables, we have to separate similarity transformations and take into consideration only those transformations under which the sum of squares of coordinates of the three particles is invariant, *i.e.* the radius of the six dimensional sphere remains constant.

Consider now a triangle, the vertices of which are determined by three particles. If we exclude the similarity transformations, two possible types of transformations are left: rotations in the ordinary three dimensional space which are described by the  $O(3)$  group, and deformations of the triangle. It can be easily seen that the deformation leads to  $SU(2)$ .

It is obvious that different forms of a deformed, non-rotating triangle can be considered as the projections onto its plane of all the possible positions of a rotating rigid triangle.

Studying both types of transformations at the same time, one can say that all the transformations of a triangle besides the similarity transformations are described by the projections onto the three-dimensional space of a rigid triangle which rotate in the four-dimensional space. This means that an arbitrary motion of three particles is equivalent to the rotation of a rigid triangle and the similarity transformations.

Let us turn our attention to a formal analogy with the Kepler problem. The planetary motion along the elliptic trajectory can be described as the projection of motion along the great circle on the four dimensional sphere onto its equatorial section. Ellipses with equal major axes are corresponding to different great circles. After carrying out the transformation of time, we can show that the Kepler motion will be described by the free motion of a point on the four-dimensional sphere, famous Fock-symmetry.<sup>41</sup> This way we arrive at the local  $O(4)$  symmetry which will also be considered.

Both the representation of the group of motions on the six-dimensional sphere and its reduction to  $SU(3)$  or  $O(4)$  involve the representation of the permutation group  $P_2(3)$ . That is why this description is extraordinary convenient for the system of three equivalent particles. Here we will restrict ourselves to this simple case. In the general case of arbitrary masses some new features will appear only when we expand the amplitudes or the wave functions of the interacting particles over the basis functions.<sup>42</sup> As it is well known, the boundary of the definition of the functions depends on the masses.

Concerning the construction of the basis, a question arises whether it is necessary to build up the basis with the help of K-polynomials of the harmonic functions of  $O(6)$ . Obviously, if the interaction between the particles is weak and their motion differs only slightly from the free one, this choice of the basis functions will be natural. If, on the contrary, the particles are strongly bounded and form an almost

rigid triangle, a basis, which do not obey the Laplace equation on the six dimensional sphere, turns out to be more convenient. As an example of such a basis, so-called B-polynomials will be constructed.

An interesting subject for discussion is given by the fifth quantum number  $\Omega$  (see Ref. 43). The introduction of this quantity becomes necessary since it is not sufficient to use the quantum numbers coming from the reduction  $O(6) = O(3) \times O(2)$ , *i.e.* the quantum numbers characterizing the rotations and permutations. In fact for a system consisting of more than three particles one has to introduce additional quantum numbers: three quantum numbers in the case of four particles, and four in the case of five particles. In the case of a system including six or more particles five new quantum numbers are necessary. It is a rather remarkable fact that for more than six particles the number of additional quantum numbers remains constant.

It is worthwhile to study also the energy spectrum of the three-particles system in the case when the triangle formed by three particles is getting rigid. The transition from the spectrum of non-interacting particles to the spectrum of the top may be investigated with the help of the presented basis.

## 2. Coordinates and Observables

The usual way of choosing the coordinates is the following. Let  $x_i$  ( $i = 1, 2, 3$ ) be the radius vectors of the three particles, and fix

$$x_1 + x_2 + x_3 = 0. \quad (1)$$

The Jacobi coordinates for equal masses will be defined as

$$\begin{aligned} \xi &= -\sqrt{\frac{3}{2}} (x_1 + x_2), & \eta &= \sqrt{\frac{1}{2}} (x_1 - x_2), \\ \xi^2 + \eta^2 &= 2x_1^2 + 2x_1x_2 + 2x_2^2 = x_1^2 + x_2^2 + x_3^2 = \rho^2. \end{aligned} \quad (2)$$

We may define similar coordinates in the momentum space as well. In that case condition (1) means that we are in the centre-of-mass frame, and  $\rho^2$  is a quantity proportional to the energy.

The quadratic form  $\rho^2$  can be understood as an invariant of the  $O(6)$  group. In fact, we are interested in the direct product  $O(3) \times O(2)$ , as we have to introduce the total angular momentum observables  $L$  and  $M$  (group  $O(3)$ ), and quantum numbers of the three-particle permutation group  $O(2)$ .

To characterize our three-particle system we need five quantum numbers. Thus the  $O(6)$  group is too large for our purposes and it is convenient to deal with  $SU(3)$  symmetry, in case of which we dispose exactly of the necessary four quantum numbers.

Let us introduce the complex vector

$$z = \xi + i\eta, \quad z^* = \xi - i\eta. \quad (3)$$

The permutation of two particles leads in terms of these coordinates to rotations in the complex  $z$ -plane:

$$P_{12} \begin{pmatrix} z \\ z^* \end{pmatrix} = \begin{pmatrix} z^* \\ z \end{pmatrix}, \quad P_{13} \begin{pmatrix} z \\ z^* \end{pmatrix} = \begin{pmatrix} e^{i\pi/3} z^* \\ e^{-i\pi/3} z \end{pmatrix}, \quad P_{23} \begin{pmatrix} z \\ z^* \end{pmatrix} = \begin{pmatrix} e^{-i\pi/3} z^* \\ e^{i\pi/3} z \end{pmatrix}. \quad (4)$$

The condition

$$\xi^2 + \eta^2 = |z|^2 = \rho^2 \quad (5)$$

gives the invariant of the group  $SU(3) \subset O(6)$ . In the following we will take  $\rho = 1$ .

The generators of  $SU(3)$  are defined, as usual:

$$A_{ik} = iz_i \frac{\partial}{\partial z_k} - iz_k^* \frac{\partial}{\partial z_i^*}. \quad (6)$$

The chain  $SU(3) \supset SU(2) \supset U(1)$  familiar from the theory of unitary symmetry of hadrons is of no use for us, because it does not contain  $O(3)$ , *i.e.* going this way we cannot introduce the angular momentum quantum numbers. Instead of that, we consider two subgroups  $O(6) \supset O(4) \sim SU(2) \times O(3)$  and  $O(6) \supset SU(3)$ . In other words, we have to separate from (5) the antisymmetric tensor-generator of the rotation group  $O(3)$

$$L_{ik} = \frac{1}{2}(A_{ik} - A_{ki}) = \frac{1}{2} \left( iz_i \frac{\partial}{\partial z_k} - iz_k \frac{\partial}{\partial z_i} + iz_i^* \frac{\partial}{\partial z_k^*} - iz_k^* \frac{\partial}{\partial z_i^*} \right). \quad (7)$$

The remaining symmetric part

$$B_{ik} = \frac{1}{2}(A_{ik} + A_{ki}) = \frac{1}{2} \left( iz_i \frac{\partial}{\partial z_k} + iz_k \frac{\partial}{\partial z_i} - iz_i^* \frac{\partial}{\partial z_k^*} - iz_k^* \frac{\partial}{\partial z_i^*} \right) \quad (8)$$

is the generator of the group of deformations of the triangle which turns out to be locally isomorphic with the rotation group. Finally, we introduce a scalar operator

$$N = \frac{1}{2i} \text{Sp } A = \frac{1}{2} \sum_k \left( z_k \frac{\partial}{\partial z_k} - z_k^* \frac{\partial}{\partial z_k^*} \right). \quad (9)$$

For characterizing our system, we choose the following quantum numbers:

$$\left. \begin{aligned} K(K+4) &- \text{eigenvalue of the Laplace operator (quadratic Casimir operator for } SU(3)); \\ L(L+1) &- \text{eigenvalue of the square of the angular momentum operator} \\ L^2 &= 4 \sum_{i>k} L_{ik}^2, \\ M &- \text{eigenvalue of } L_3 = 2L_{12}, \\ \nu &- \text{eigenvalue of } N. \end{aligned} \right\}. \quad (10)$$

Although the generator (9) is not a Casimir operator of  $SU(3)$ , the representation might be characterized by means of its eigenvalue, because, as it can be seen, the eigenvalue of the Casimir operator of third order can be written as a combination of  $K$  and  $\nu$ . If the harmonic function belongs to the representation  $(p, q)$  of  $SU(3)$ , then

it is the eigenfunction of  $\Delta$  and  $N$  with eigenvalues  $K(K+4)$  and  $\nu$  respectively, where  $K = p + q$  and  $\nu = p - q$ .

The fifth quantum number is not included in any of the considered subgroups, we have to take it from  $O(6)$ . We define it as the eigenvalue of

$$\Omega = \sum_{ikl} L_{ik} B_{kl} L_{li} = \text{Sp } LBL. \quad (11)$$

This cubic generator was first introduced by Racah.<sup>44</sup> Its physical meaning will be discussed later.

### 2.1. Parametrization of a complex sphere

Dealing with a three-particle system, we have to introduce coordinates which refer explicitly to the moving axes. One of the possible parametrizations of the vectors  $z$  and  $z^*$  is the following:

$$\begin{aligned} z &= \frac{1}{\sqrt{2}} e^{-i\lambda/2} \left( e^{ia/2} l_1 + i e^{-ia/2} l_2 \right), \\ z^* &= \frac{1}{\sqrt{2}} e^{i\lambda/2} \left( e^{-ia/2} l_1 - i e^{ia/2} l_2 \right), \\ |z|^2 &= 1, \quad l_1^2 = l_2^2 = 1, \quad l_1 l_2 = 0. \end{aligned} \quad (12)$$

In terms of these variables the (diagonal) moment of inertia has the following components:

$$\sin^2 \left( \frac{a}{2} - \frac{\pi}{4} \right), \quad \cos^2 \left( \frac{a}{2} - \frac{\pi}{4} \right), \quad l. \quad (13)$$

The three orthogonal unit vectors  $l_1, l_2$  and  $l = l_1 \times l_2$  form the moving system of coordinates. Their orientation to the fixed coordinate system can be described with the help of the Euler angles  $\varphi_1, \theta, \varphi_2$ :

$$I_1 = \left\{ -\sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2 \cos \theta \right. \quad (14)$$

$$\left. -\sin \varphi_1 \cos \varphi_2 - \cos \varphi_1 \sin \varphi_2 \cos \theta; -\cos \varphi_1 \sin \theta \right\},$$

$$I_2 = \left\{ -\cos \varphi_1 \sin \varphi_2 - \sin \varphi_1 \cos \varphi_2 \cos \theta; \right. \\ \left. -\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2 \cos \theta; \sin \varphi_1 \sin \theta \right\},$$

$$I = \left\{ -\cos \varphi_2 \sin \theta; \sin \varphi_2 \sin \theta; -\cos \theta \right\}. \quad (15)$$

In the following it will be simpler to introduce a new angle

$$a = \alpha - \frac{\pi}{2} \quad (16)$$

and work with the vectors:

$$\begin{aligned} z &= e^{-i\lambda/2} \left( \cos \frac{a}{2} l_+ + i \sin \frac{a}{2} i_- \right), \quad z^* = e^{-i\lambda/2} \left( \cos \frac{a}{2} l_- - \sin \frac{a}{2} l_+ \right), \\ l_+ &= \frac{1}{\sqrt{2}} (l_1 + i l_2), \quad l_- = \frac{1}{\sqrt{2}} (l_1 - i l_2). \end{aligned} \quad (17)$$

Vectors  $l_+$  and  $l_-$  have the obvious properties

$$l_+^2 = l_-^2 = 0, \quad l_0 = (l_+ \times l_-) = -il, \quad l_+ l_- = 1, \quad l_+^* = l_- . \quad (18)$$

Let us turn our attention to the fact that the components of  $l_+$  and  $l_-$  may be expressed in terms of the Wigner D-functions, defined as

$$D_{mn}^l(\varphi_1 \theta \varphi_2) = e^{-i(m\varphi_1 + n\varphi_2)} P_{mn}^l(\cos \theta) \quad (19)$$

in the following way:

$$\begin{aligned} l_+ &= \{D_{1-1}^1(\varphi_1 \theta \varphi_2); D_{10}^1(\varphi_1 \theta \varphi_2); D_{11}^1(\varphi_1 \theta \varphi_2)\}, \\ l_0 &= \{D_{0-1}^1(\varphi_1 \theta \varphi_2); D_{00}^1(\varphi_1 \theta \varphi_2); D_{01}^1(\varphi_1 \theta \varphi_2)\}, \\ l_- &= \{D_{-1-1}^1(\varphi_1 \theta \varphi_2); D_{-10}^1(\varphi_1 \theta \varphi_2); D_{-11}^1(\varphi_1 \theta \varphi_2)\}, \end{aligned} \quad (20)$$

These equations demonstrate the possibility to construct the Wigner functions from the unit vectors corresponding to the moving coordinate system, in a way similar to the construction of spherical harmonics from the unit vectors of the fixed coordinate system. However, we see that the traditional parametrization of the vectors  $l_i$  which we have introduced is not fortunate; it would be much more aesthetical to go over to a parametrization in which

$$D_{mn}^1(\varphi_1 \theta \varphi_2) = l_m k_n ,$$

where  $l_m$  and  $k_n$  are unit vectors of the moving and fixed coordinate systems, respectively. Yet, so far we will not change the parametrization.

The vectors  $z$  and  $z^*$  can be written as

$$\begin{aligned} z_M &= \sum_{M'=\pm 1/2} D_{1/2, M'}^{1/2}(\lambda, a, 0) D_{2M', M}^1(\varphi_1 \theta \varphi_2), \\ z_M^* &= D_{-1/2, -1/2}^{1/2}(\lambda, a, 0) D_{-1, M}^1(\varphi_1 \theta \varphi_2) - D_{-1/2, 1/2}^{1/2}(\lambda, a, 0) D_{1, M}^1(\varphi_1 \theta \varphi_2). \end{aligned} \quad (21)$$

## 2.2. The Laplace operator

We have now to write the operators, the eigenvalues of which we are looking for. Let us first construct the Laplace operator. We could do that by a straightforward calculation of  $\Delta = |A_{ik}|^2$ , but we choose a simpler way. We calculate

$$dz = -\frac{i}{2} z d\lambda + \frac{1}{2} e^{-i\lambda} (l \times z^*) da - (d\omega \times z). \quad (22)$$

This rather simple expression is obtained by introducing the infinitesimal rotation  $d\omega$ . Its projections onto the fixed coordinate  $k_1 = (1, 0, 0)$ ,  $k_2 = (0, 1, 0)$ ,  $k_3 = (0, 0, 1)$  given in terms of the Euler angles are well known:

$$\begin{aligned} d\omega_1 &= \cos \varphi_2 \sin \theta d\varphi_1 - \sin \varphi_2 d\theta, \\ d\omega_2 &= -\sin \varphi_2 \sin \theta d\varphi_1 - \cos \varphi_2 d\theta, \\ d\omega_3 &= \cos \theta d\varphi_1 + d\varphi_2. \end{aligned} \quad (23)$$



This provides

$$\begin{aligned}\frac{\partial}{\partial\omega_1} &= \cos\varphi_2 \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi_1} - \cos\varphi_2 \frac{\partial}{\partial\varphi_2} - \sin\varphi_2 \frac{\partial}{\partial\theta}, \\ \frac{\partial}{\partial\omega_2} &= -\sin\varphi_2 \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi_1} + \sin\varphi_2 \operatorname{ctg}\theta \frac{\partial}{\partial\varphi_2} - \cos\varphi_2 \frac{\partial}{\partial\theta}, \\ \frac{\partial}{\partial\omega_3} &= \frac{\partial}{\partial\varphi_2},\end{aligned}\tag{24}$$

and the permutation relations

$$\left[\frac{\partial}{\partial\omega_1}, \frac{\partial}{\partial\omega_2}\right] = \frac{\partial}{\partial\omega_3}, \quad \left[\frac{\partial}{\partial\omega_2}, \frac{\partial}{\partial\omega_3}\right] = \frac{\partial}{\partial\omega_1}, \quad \left[\frac{\partial}{\partial\omega_3}, \frac{\partial}{\partial\omega_1}\right] = \frac{\partial}{\partial\omega_2}.\tag{25}$$

The effect of this operator on an arbitrary vector  $A$  is

$$\frac{\partial}{\partial\omega_i} A = \omega_i \times A;\tag{26}$$

Here  $\omega_i$  is a vector of the length  $\omega_i$ , directed along the  $i$  axis. The expression can be checked using the perturbation relation.

Let us determine now the rotation around the moving axes:

$$d\Omega_i = l_i d\omega.\tag{27}$$

In an explicit form  $\partial/\partial\Omega_i$  can be written

$$\begin{aligned}\frac{\partial}{\partial\Omega_1} &= \cos\varphi_1 \operatorname{ctg}\theta \frac{\partial}{\partial\varphi_1} - \cos\varphi_1 \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi_2} + \sin\varphi_1 \frac{\partial}{\partial\theta}, \\ \frac{\partial}{\partial\Omega_2} &= -\sin\varphi_1 \operatorname{ctg}\theta \frac{\partial}{\partial\varphi_1} + \sin\varphi_1 \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi_2} + \cos\varphi_1 \frac{\partial}{\partial\theta}, \\ \frac{\partial}{\partial\Omega_3} &= -\frac{\partial}{\partial\varphi_1}.\end{aligned}\tag{28}$$

The minus sign in the third component reflects our choice of normalization of the  $D$ -function with a minus in the exponent (19).

The permutation relations for the operators  $\partial/\partial\Omega_i$  are

$$\left[\frac{\partial}{\partial\Omega_1}, \frac{\partial}{\partial\Omega_2}\right] = -\frac{\partial}{\partial\Omega_3}, \quad \left[\frac{\partial}{\partial\Omega_2}, \frac{\partial}{\partial\Omega_3}\right] = -\frac{\partial}{\partial\Omega_1}, \quad \left[\frac{\partial}{\partial\Omega_3}, \frac{\partial}{\partial\Omega_1}\right] = -\frac{\partial}{\partial\Omega_2}.\tag{29}$$

The effect on  $A$  is defined, correspondingly, as

$$\frac{\partial}{\partial\Omega} A = -\Omega_i \times a,\tag{30}$$

which differs from (26) by the sign, as a consequence of the different signs in the permutation relations (25) and (29).

From (22) we obtain

$$\begin{aligned}ds^2 &= dz dz^* = g_{ik} x^i x^k \\ &= \frac{1}{4} da^2 + \frac{1}{4} d\lambda^2 + \frac{1}{2} d\Omega_1^2 + \frac{1}{2} d\Omega_2^2 + d\Omega_3^2 - \sin a d\Omega_1 d\Omega_2 - \cos a d\Omega_3 d\lambda.\end{aligned}\tag{31}$$

This expression determines the components of the metric tensor  $q_{ik}$ , and it becomes easy to calculate the Laplace operator

$$\begin{aligned}\Delta' &= \frac{1}{4} \Delta = \frac{1}{4} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} g^{ik} \sqrt{g} \frac{\partial}{\partial x^k} = \\ &= \frac{\partial^2}{\partial a^2} + 2 \operatorname{ctg} 2a \frac{\partial}{\partial a} + \frac{1}{\sin^2 a} \left( \frac{\partial^2}{\partial \lambda^2} + \cos a \frac{\partial^2}{\partial \lambda \partial \Omega_3} + \frac{1}{4} \frac{\partial^2}{\partial \Omega_3^2} \right) + \\ &\quad + \frac{1}{2 \cos^2 a} \left[ \frac{\partial^2}{\partial \Omega_1^2} + \sin a \left( \frac{\partial^2}{\partial \Omega_1 \partial \Omega_2} + \frac{\partial^2}{\partial \Omega_2 \partial \Omega_1} \right) + \frac{\partial^2}{\partial \Omega_2^2} \right].\end{aligned}\quad (32)$$

If  $\Phi$  is the eigenfunction of  $\Delta'$ , corresponding to a definite representation of  $SU(3)$ , then

$$\Delta' \Phi = -\frac{1}{4} K(K+4) \Phi = -\frac{K}{2} \left( \frac{K}{2} + 2 \right) \Phi \quad (33)$$

and

$$N \Phi = \nu \Phi, \quad N = i \frac{\partial}{\partial \lambda} \quad (34)$$

has to be fulfilled.

Expressing (32) in terms of the Euler angles, we get the Laplace operator in the form obtained in Ref. 40:

$$\begin{aligned}\Delta' &= \Delta_a - \operatorname{tg} a \frac{\partial}{\partial a} + \frac{1}{2 \cos^2 a} \left( \Delta_\theta - \frac{\partial^2}{\partial \varphi_1^2} \right) - \frac{\sin a}{2 \cos^2 a} \times \\ &\quad \times \left[ \cos 2\varphi_1 \left( \frac{1 + \cos^2 \theta}{\sin^2 \theta} \frac{\partial}{\partial \varphi_1} - 2 \frac{\cos \theta}{\sin^2 \theta} \frac{\partial}{\partial \varphi_2} - 2 \operatorname{ctg} \theta \frac{\partial^2}{\partial \varphi_1 \partial \theta} + 2 \frac{1}{2 \sin \theta} \frac{\partial^2}{\partial \varphi_2 \partial \theta} \right) \right. \\ &\quad \left. + \sin 2\varphi_1 \left( \Delta_\theta - \frac{\partial^2}{\partial \varphi_1^2} - 2 \frac{\partial^2}{\partial \theta^2} \right) \right],\end{aligned}\quad (35)$$

where  $\Delta_a$  and  $\Delta_\theta$  are the Laplace operators

$$\begin{aligned}\Delta_a &= \frac{\partial^2}{\partial a^2} + \operatorname{ctg} a \frac{\partial}{\partial a} + \frac{1}{\sin^2 a} \left( \frac{\partial^2}{\partial \lambda^2} + \cos a \frac{\partial^2}{\partial \lambda \partial \Omega_3} + \frac{1}{4} \frac{\partial^2}{\partial \Omega_3^2} \right), \\ \Delta_\theta &= \frac{\partial^2}{\partial \theta^2} + \operatorname{ctg} \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial \varphi_1^2} - 2 \cos \theta \frac{\varphi^2}{\partial \varphi_1 \partial \varphi_2} + \frac{\partial^2}{\partial \varphi_2^2} \right)\end{aligned}\quad (36)$$

of the  $O(3)$  group. The Laplace operator (35) differs from that calculated in Ref. 40 by the parametrization. They are connected, however, by a unitary transformation.

### 2.3. Calculation of the generators $L_{ik}$ and $B_{ik}$

To obtain the generators directly from  $dz$ , we have to invert a  $5 \times 5$  matrix in the case of a three-particle system. That requires rather a long calculation, which is getting hopeless for a larger number of particles. Instead of performing the straightforward calculation, we get the wanted expressions in the following way. Let us first consider  $L_{ik}$ , or rather one of its components, e.g.  $L_{12}$ . We introduce a parameter  $\sigma_{ik}$  which

defines the motion along the particular trajectory which corresponds to the action of the operator  $L_{ik}$ . Thus, formally we can write

$$L_{12} = \frac{1}{2} \left( iz_1 \frac{\partial}{\partial z_2} - iz_2 \frac{\partial}{\partial z_1} + iz_1^* \frac{\partial}{\partial z_2^*} - iz_2^* \frac{\partial}{\partial z_1^*} \right) \equiv \frac{\partial}{\partial \sigma_{12}}. \quad (37)$$

Acting with  $L_{12}$  on the vectors  $z$  and  $z^*$

$$L_{12} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -iz_2 \\ iz_1 \\ 0 \end{pmatrix}, \quad L_{12} \begin{pmatrix} z_1^* \\ z_2^* \\ z_3^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -iz_2^* \\ iz_1^* \\ 0 \end{pmatrix} \quad (38)$$

we see that  $\sigma_{12}$  has to be imaginary. From (38) we get

$$\begin{aligned} z L_{12} z &= 0, & z^* L_{12} z^* &= 0, \\ z^* L_{12} z &= \frac{i}{2} (z \times z^*)_3, & l L_{12} z &= -\frac{i}{2} (l \times z)_3. \end{aligned} \quad (39)$$

Making use of the expression (22) for  $dz$ , we can write

$$\begin{aligned} L_{12} z &= \frac{\partial z}{\partial \sigma_{12}} = -\frac{i}{2} z \frac{d\lambda}{d\sigma_{12}} + \frac{1}{2} e^{-i\lambda} (l \times z^*) \frac{\partial a}{\partial \sigma_{12}} - \left( \frac{d\omega}{d\sigma_{12}} \times z \right), \\ L_{12} z^* &= \frac{\partial z^*}{\partial \sigma_{12}} = \frac{i}{2} z^* \frac{d\lambda}{d\sigma_{12}} + \frac{1}{2} e^{i\lambda} (l \times z) \frac{\partial a}{\partial \sigma_{12}} - \left( \frac{d\omega}{d\sigma_{12}} \times z^* \right), \end{aligned} \quad (40)$$

(here  $-iz/2$  is  $dz/d\lambda$  etc.). We use Eq. (39). Substituting in Eq. (40)

$$\begin{aligned} (l \times z^*) &= ie^{i\lambda/2} \left( \cos \frac{a}{2} l_- + \sin \frac{a}{2} l_+ \right), \\ (l \times z) &= -ie^{-i\lambda/2} \left( \cos \frac{a}{2} l_+ - i \sin \frac{a}{2} l_- \right); \\ z^2 &= ie^{-i\lambda} \sin a, \quad z^{*2} = -ie^{i\lambda} \sin a, \end{aligned} \quad (41)$$

we obtain from Eq. (39)

$$\frac{\partial a}{\partial \sigma_{12}} = \frac{d\lambda}{d\sigma_{12}} = 0. \quad (42)$$

Similarly, (39) gives

$$\frac{d\Omega_3}{d\sigma_{12}} = -\frac{i}{2} l^{(3)}, \quad \frac{d\Omega_2}{d\sigma_{12}} = \frac{i}{2} l_2^{(3)}, \quad \frac{d\Omega_1}{d\sigma_{12}} = -\frac{i}{2} l_l^{(3)}, \quad (43)$$

where  $l_i^{(k)}$  stands for the  $k$ -component of vector  $l_i$ . Thus we obtain

$$L_{12} = -\frac{i}{2} \left[ l_1^{(3)} \frac{\partial}{\partial \Omega_1} + l_2^{(3)} \frac{\partial}{\partial \Omega_2} + l_3^{(3)} \frac{\partial}{\partial \Omega_3} \right] = -\frac{i}{2} \frac{\partial}{\partial \omega_3}. \quad (44)$$

and

$$\begin{aligned} L_{23} &= -\frac{i}{2} \left[ l_1^{(1)} \frac{\partial}{\partial \Omega_1} + l_2^{(1)} \frac{\partial}{\partial \Omega_2} + l_3^{(1)} \frac{\partial}{\partial \Omega_3} \right] = -\frac{i}{2} \frac{\partial}{\partial \omega_1}, \\ L_{31} &= -\frac{i}{2} \left[ l_1^{(2)} \frac{\partial}{\partial \Omega_1} + l_2^{(2)} \frac{\partial}{\partial \Omega_2} + l_3^{(2)} \frac{\partial}{\partial \Omega_3} \right] = -\frac{i}{2} \frac{\partial}{\partial \omega_2}. \end{aligned} \quad (45)$$

Introducing the notations

$$L_1 = 2L_{23}, \quad L_2 = 2L_{31}, \quad L_3 = 2L_{12}, \quad (46)$$

we can write the general expression for the angular momentum operator

$$L_k = -i \left[ l_1^{(k)} \frac{\partial}{\partial \Omega_1} + l_2^{(k)} \frac{\partial}{\partial \Omega_2} + l_3^{(k)} \frac{\partial}{\partial \Omega_3} \right]. \quad (47)$$

It satisfies the commutation relations

$$[L_1, L_2] = -iL_3, \quad [L_2, L_3] = -iL_1, \quad [L_3, L_1] = -iL_2. \quad (48)$$

The square of the angular momentum operator is

$$L^2 = \left( \frac{\partial^2}{\partial \Omega_1^2} + \frac{\partial^2}{\partial \Omega_2^2} + \frac{\partial^2}{\partial \Omega_3^2} \right) = \Delta_\theta. \quad (49)$$

Let us now turn our attention to the operator  $B_{ik}$ . We consider

$$B_{12} = \frac{1}{2} \left( iz_1 \frac{\partial}{\partial z_2} + iz_2 \frac{\partial}{\partial z_1} - iz_1^* \frac{\partial}{\partial z_2^*} - iz_2^* \frac{\partial}{\partial z_1^*} \right) \equiv \frac{\partial}{\partial \beta_{12}}. \quad (50)$$

From the action of  $B_{12}$  on  $z$  and  $z^*$

$$B_{12} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} iz_2 \\ iz_1 \\ 0 \end{pmatrix}, \quad B_{12} \begin{pmatrix} z_1^* \\ z_2^* \\ z_3^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -z_2^* \\ -z_1^* \\ 0 \end{pmatrix}, \quad (51)$$

it is obvious, that  $\beta_{12}$  is real. We make use of the conditions

$$\begin{aligned} z B_{12} z &= iz_1 z_2, & z^* B_{12} z^* &= -iz_1^* z_2^*, \\ z^* B_{12} z &= \frac{i}{2} (z_1^* z_2 + z_1 z_2^*), & l B_{12} z &= \frac{i}{2} (l^{(1)} z_2 + l^{(2)} z_1) \end{aligned} \quad (52)$$

and of (22) and (41).

Let us introduce the notation

$$b_{ik}^{(lm)} = \frac{1}{2} \left( l_i^{(l)} l_k^{(m)} + l_i^{(m)} l_k^{(l)} \right). \quad (53)$$

Then, following a procedure similar to that in the case of  $L_{ik}$ , we obtain from Eq. (52)

$$\begin{aligned} \frac{da}{d\beta_{12}} &= b_{11}^{(12)} - b_{22}^{(12)}, \\ \frac{d\lambda}{d\beta_{12}} &= \left( b_{11}^{(12)} + b_{22}^{(12)} \right) - 2b_{12}^{(12)} \frac{1}{\sin a}. \end{aligned} \quad (54)$$

Equations (52) lead to

$$\begin{aligned} \frac{d\Omega_1}{d\beta_{12}} &= -b_{23}^{(12)} \operatorname{tg} a - b_{13}^{(12)} \frac{1}{\cos a}, \\ \frac{d\Omega_2}{d\beta_{12}} &= -b_{23}^{(12)} \frac{1}{\cos a} - b_{13}^{(12)} \operatorname{tg} a, \\ \frac{d\Omega_3}{d\beta_{12}} &= -b_{12}^{(12)} \operatorname{ctg} a. \end{aligned} \quad (55)$$

Thus the expression for  $B_{12}$  can be written as

$$\begin{aligned} B_{12} = & \left( b_{11}^{(12)} - b_{22}^{(12)} \right) \frac{\partial}{\partial a} - \left( b_{11}^{(12)} + b_{22}^{(12)} \right) \frac{\partial}{\partial \lambda} - 2b_{12}^{(12)} \left( \frac{1}{\sin a} \frac{\partial}{\partial \lambda} + \frac{1}{2} \text{ctg} a \frac{\partial}{\partial \Omega_3} \right) \\ & - \left( b_{13}^{(12)} \frac{1}{\cos a} + b_{23}^{(12)} \text{tg} a \right) \frac{\partial}{\partial \Omega_1} - \left( b_{13}^{(12)} \text{tg} a + b_{23}^{(12)} \frac{1}{\cos a} \right) \frac{\partial}{\partial \Omega_2}. \end{aligned} \quad (56)$$

The generator  $B_{ik}$  of the deformation group of the triangle obtains the form

$$\begin{aligned} B_{ik} = & \left( b_{11}^{(ik)} - b_{22}^{(ik)} \right) \frac{\partial}{\partial a} - \left( b_{11}^{(ik)} + b_{22}^{(ik)} \right) \frac{\partial}{\partial \lambda} - 2b_{12}^{(ik)} \left( \frac{1}{\sin a} \frac{\partial}{\partial \lambda} + \frac{1}{2} \text{ctg} a \frac{\partial}{\partial \Omega_3} \right) \\ & - b_{13}^{(ik)} \left( \text{tg} a \frac{\partial}{\partial \Omega_2} + \frac{1}{\cos a} \frac{\partial}{\partial \Omega_1} \right) - b_{23}^{(ik)} \left( \text{tg} a \frac{\partial}{\partial \Omega_1} + \frac{1}{\cos a} \frac{\partial}{\partial \Omega_2} \right). \end{aligned} \quad (57)$$

Acting in the space of polynomials which include only  $z$  (and not  $z^*$ ), the following identity appears:

$$ie^{i\alpha} \frac{\partial}{\partial \Omega_1} = \frac{\partial}{\partial \Omega_2}. \quad (58)$$

Thus in the space of polynomials of  $z$   $B_{ik}$  might be written as

$$\begin{aligned} B_{ik} = & \left( b_{11}^{(ik)} - b_{22}^{(ik)} \right) \frac{\partial}{\partial a} - \left( b_{11}^{(ik)} + b_{22}^{(ik)} \right) \frac{\partial}{\partial \lambda} + 2 \frac{\partial}{\partial \lambda} \delta_{ik} \\ & - 2b_{12}^{(ik)} \left( \frac{1}{\sin a} \frac{\partial}{\partial \lambda} + \frac{1}{2} \text{ctg} a \frac{\partial}{\partial \Omega_3} \right) - ib_{23}^{(ik)} \frac{\partial}{\partial \Omega_1} + ib_{13}^{(ik)} \frac{\partial}{\partial \Omega_2}. \end{aligned} \quad (59)$$

Let us present also the permutation expressions

$$\begin{aligned} [B_{ik}, B_{jl}] &= \frac{1}{2} (L_{il} \delta_{kj} - L_{jk} \delta_{il}) + \frac{i}{2} (L_{ij} \delta_{kl} - L_{lk} \delta_{ij}), \\ [B_{ik}, L_{jl}] &= \frac{i}{2} (B_{il} \delta_{kj} - B_{jk} \delta_{il}) - \frac{i}{2} (B_{ij} \delta_{kl} - B_{ik} \delta_{lj}). \end{aligned} \quad (60)$$

In particular,

$$\begin{aligned} [B_{12}, B_{11}] &= -iL_{12}, \quad [B_{12}, B_{22}] = iL_{12}, \quad [B_{11}, L_{12}] = iB_{12}, \\ [B_{22}, L_{12}] &= -iB_{12}, \quad [B_{12}, L_{12}] = -\frac{i}{2}(B_{11} - B_{22}). \end{aligned} \quad (61)$$

From this it follows that  $L_{12}$ ,  $B_{12}$  and  $1/2(B_{11} - B_{22})$  form the  $SU(2)$  subgroup.

#### 2.4. The cubic operator $\Omega$

Operators  $H_+$  and  $H_-$  are the usual raising and lowering operators in  $SU(2)$  taken at the value of the second Euler angle  $-2\Omega_3 = 2\varphi_1 = 0$

$$\begin{aligned} H_+ &= \frac{1}{\sqrt{2}} \left[ \frac{\partial}{\partial a} + i \frac{1}{\sin a} \frac{\partial}{\partial \lambda} + \frac{i}{2} \text{ctg} a \frac{\partial}{\partial \Omega_3} \right], \\ H_- &= \frac{1}{\sqrt{2}} \left[ \frac{\partial}{\partial a} - i \frac{1}{\sin a} \frac{\partial}{\partial \lambda} - \frac{i}{2} \text{ctg} a \frac{\partial}{\partial \Omega_3} \right], \end{aligned} \quad (62)$$

$\Omega$  can be written in the form

$$\begin{aligned} \Omega = \sum_{i,j,k} L_{ij} L_{jk} B_{ki} = & -\frac{1}{4} \left\{ \sqrt{2} \left( -\frac{\partial^2}{\partial \Omega_+^2} H_+ + \frac{\partial^2}{\partial \Omega_-^2} H_- \right) + \frac{\partial^2}{\partial \Omega_3^2} \frac{\partial}{\partial \lambda} + \Delta_\theta \frac{\partial}{\partial \lambda} \right. \\ & \left. - \frac{1}{\cos a} \left( \Delta_\theta - \frac{\partial^2}{\partial \Omega_3^2} + \frac{1}{2} \right) \frac{\partial}{\partial \Omega_3} + \operatorname{tg} a \left[ i \left( \frac{\partial^2}{\partial \Omega_+^2} - \frac{\partial^2}{\partial \Omega_-^2} \right) \frac{\partial}{\partial \Omega^2} - \frac{3}{2} \left( \frac{\partial^2}{\partial \Omega_+^2} + \frac{\partial^2}{\partial \Omega_-^2} \right) \right] \right\}. \end{aligned} \quad (63)$$

The operator  $\Omega$  has a simple meaning in the classical approximation. Changing the derivative to the velocity and denoting  $\xi = p$  and  $\eta = q$ , we obtain

$$\frac{1}{2} \Omega = (\xi L)(qL) - (\eta L)(pL). \quad (64)$$

The time derivative of this operator is, obviously, zero. If we direct the axis  $z$  along  $L$  and introduce two two-dimensional vectors in the space of permutation

$$x = (\xi_z, \eta_z) \text{ and } y = (p_z, q_z), \quad (65)$$

then (I.86) can be written as

$$\frac{1}{2} \Omega = (x \times y)_3. \quad (66)$$

The operator has the form of the third component of the momentum in the permutation space. Hence, the symmetry of the problem becomes obvious: it is spherical in the coordinate space, and axial in the permutation space.

We do not need the eigenvalues of this operator at small  $K$  and  $\nu$  values when the degeneracy is small. Indeed, at a given  $K$  and  $\nu$  the number of states is determined by the usual  $SU(3)$  expression

$$n(K, \nu) = \frac{1}{8} (K+2)(K+2-2\nu)(K+2+2\nu). \quad (67)$$

Summing up this formula over  $2\nu$  from  $-K$  to  $K$ , we obtain the well-known expression:<sup>39</sup>

$$n(K) = \frac{(K+3)(K+2)^2(K+1)}{12}. \quad (68)$$

The terms for maximal degeneracies go from  $\nu = 0$  in the case of even  $K$  or from  $\nu = 1/2$  in the case of odd  $K$ .

$$n(K, 0) = \begin{cases} \frac{1}{8} (K+2)^3, & K - \text{ odd } , \\ \frac{1}{8} (K+1)(K+2)(K+3), & K - \text{ even } . \end{cases} \quad (69)$$

### 2.5. Solution of the eigenvalue problem

$$\Phi_M^L = \sum_{\lambda} \sum_{M'=-\lambda}^{\lambda} a(\Lambda, M') a(\Lambda, M') D_{\nu, M'}^{\Lambda}(\lambda, a, 0) D_{2M', M}^L(\varphi_1, \theta, \varphi_2). \quad (70)$$

Let us finally consider a few special cases of the solution. As it is discussed by Dragt,<sup>45</sup> in the low-dimensional representations of  $SU(3)$  ( $L = 0, 1$ ) the  $\Omega$  is not needed. Indeed, in the case of  $L = 0$  the Laplace operator obtains the form

$$\Delta = \frac{\partial^2}{\partial a^2} + 2\text{ctg}2a \frac{\partial}{\partial a} + \frac{1}{\sin^2 a} \frac{\partial^2}{\partial \lambda^2}. \quad (71)$$

Obviously, the eigenfunction will be the following

$$\Phi_0 = D_{\nu, 0}^{\Lambda}(\lambda, a, 0), \quad (72)$$

which obeys the equation

$$\Delta \Phi_0 = -\Lambda(\Lambda + 1)\Phi_0, \quad \lambda = 0, 1, \dots \quad (73)$$

This solution demonstrates clearly the  $SU(2)$  nature of a non-rotating triangle.

In the case of  $L = 1$  the solutions are

$$\begin{aligned} z_M &= \sum_{M'=\pm 1/2} D_{1/2, M'}^{1/2}(\lambda, a, 0) D_{2M', M}^1(\varphi_1, \theta, \varphi_2), \\ z_M^* &= D_{-1/2, -1/2}^{1/2}(\lambda, a, 0) D_{-1, M}^1(\varphi_1 \theta \varphi_2) - D_{-1/2, 1/2}^{1/2}(\lambda, a, 0) D_{1, M}^1(\varphi_1 \theta \varphi_2), \end{aligned} \quad (74)$$

fulfilling the Laplace equation with the value  $K = 1$ . Simultaneously  $z_M$  obeys the equations

$$\begin{aligned} L^2 z_M &= -2z_M, \quad L_3 z_M = -M z_M, \quad M = -1, 0, 1, \\ \Delta_a z_M &= -\frac{3}{4} z_M, \quad N z_M = \frac{1}{2} z_M, \quad \Omega z_M = -\frac{3}{4} i z_M, \end{aligned} \quad (75)$$

and, accordingly,  $z_M^*$  obeys

$$\begin{aligned} L^2 z_M^* &= -2z_M^*, \quad L_3 z_M^* = -M z_M^*, \quad \Delta_a z_M^* = -\frac{3}{4} z_M^*, \\ N z_M^* &= -\frac{1}{2} z_M^*, \quad \Omega z_M^* = \frac{3}{4} i z_M^*. \end{aligned} \quad (76)$$

### 3. Eigenfunctions in the Three-Body Problem

The investigation of the three-particle system leads to the construction of basis functions in the form of the so-called K-polynomials, *i.e.* harmonic polynomials in the six-dimensional space. In order to make it possible to work with such functions which would be a natural generalization of the usual spherical functions given on the two-dimensional sphere, it is necessary to find the total system of solutions for the Laplace equation on the five-dimensional sphere.

In the previous section a method for calculating the generators of the group of motion on the five-dimensional sphere was found, and the corresponding system of commuting operators was constructed.

A somewhat unexpected difficulty of the task is due to the fact that the functions realize a representation of the permutation group of three particles and at the same time they are eigenfunctions of the operator of the momentum. If we do not require a permutation symmetry of the eigenfunction, then, obviously, the problem can be easily solved. In this case the simplest way of finding the solution is via the “tree”-function method.<sup>42</sup> The obtained eigenfunctions (the basis functions) can be characterized by five quantum numbers:

$$K, j_1, M_1, j_2, M_2, \quad (77)$$

where  $K$  – is the general order of the polynomial,  $j_1, M_1, j_2, M_2$  are the momenta and their projections corresponding to  $\xi$  and  $\eta$ . Instead of  $j_1$  and  $j_2$ , we can, of course, introduce the total momentum  $J$ .

In Ref. 31 a system of functions was built up with certain permutation symmetries. This system was characterized by the quantum numbers

$$K, J, M, \nu, \Omega. \quad (78)$$

The last two of them do not coincide with the quantum numbers corresponding to the “tree”. The general solution is of the form

$$\Phi_{M, \nu}^J = \sum a_\nu(\Lambda, M') D_{\nu, M'}^\Lambda(\lambda, a, 0) D_{2M', M}^J(\varphi_1, \theta, \varphi_2). \quad (79)$$

The meaning of the variables will be explained below; the coefficients  $a_\nu(\Lambda, M')$  had to be determined so that the function (79) satisfied the Laplace equation on the five-dimensional sphere and also the equation for the eigenvalues of the operator  $\Omega$ . Although the set of equations is simple enough to solve it in any concrete case, we were not able to find a general solution.

Let us try to solve the problem in a different way. We will carry out a transition from the total set of functions constructed by the “tree”-method to the  $K$ -harmonics. In this approach the “tree”-functions will first be transformed into a system with a given total momentum, *i.e.* to that with the quantum numbers

$$K, J, M, j_1, j_2. \quad (80)$$

After that we make a transition to the quantum numbers

$$K, J, M, \nu, (j_1, j_2). \quad (81)$$

We shall see that this can be done by a simple Fourier transformation. In fact the pair  $(j_1, j_2)$  is not a real quantum number in the sense that functions with different  $(j_1, j_2)$  pairs do not form an orthonormal system, just remind the genealogy of the functions. Let us underline here that  $j_1$  and  $j_2$  ceased to be eigenvalues after we turned to the Fourier components.



In order to solve our problem, it remains to construct the sum

$$C_{(j_1 j_2)} \Psi_{KJM}^{(j_1 j_2)}, \quad (82)$$

where  $(j_1 j_2)$  run through all values of momentum pairs which can be used for building the total momentum  $J \leq j_1 + j_2 \leq K$ .

We will demonstrate here how the set of functions (81) can be produced.

So far we were not able to find the set of eigenfunctions of the operator  $\Omega$  in a closed form. In Ref. 43 an algorithm was given for calculating such functions in the form of series. However, the order of the corresponding equation grows with the growth of the eigenvalue, and, hence, the problem of obtaining the solution in a general form remains open.

The calculations below lead to such a form of the “tree”-functions for which the Fourier transformation becomes simple; formally, we just show how to carry out the Fourier transformation of the “tree”-function.

### 3.1. Coordinates and parametrization

Let us remind the determination of the coordinates used in the previous section. The three radius vectors  $x_i$  ( $i = 1, 2, 3$ ) which are connected by the condition  $x_1 + x_2 + x_3 = 0$  form the the Jacobi-coordinates  $\xi$  and  $\eta$  for equal masses:

$$\begin{aligned} \xi &= -\sqrt{\frac{3}{2}}(x_1 + x_2), \quad \eta = \sqrt{\frac{1}{2}}(x_1 - x_2), \quad \xi^2 + \eta^2 = \rho^2, \\ z &= \xi + i\eta, \quad z^* = \xi - i\eta. \end{aligned} \quad (83)$$

Here  $\rho$  is the radius of a five-dimensional sphere; for simplicity, we suppose it to be unity.

Let us consider a triangle in the vertices of which three particles are placed. The situation of this triangle in the space is determined by the complex vectors  $\vec{l}_+$  and  $\vec{l}_-$  which, together with the third vector  $\vec{l}_0 = [\vec{l}_+ \times \vec{l}_-]$ , form a moving coordinate system. They satisfy the usual conditions

$$l_+^2 = l_-^2 = 0, \quad (\vec{l}_+ \vec{l}_-) = 1 \quad (84)$$

The vectors  $\vec{l}_+$  and  $\vec{l}_-$  are related to  $z$  and  $z^*$  by the expressions

$$\begin{aligned} z &= e^{-i\lambda/2} \left( \cos \frac{a}{2} l_+ + i \sin \frac{a}{2} l_- \right), \\ z^* &= e^{i\lambda/2} \left( \cos \frac{a}{2} l_- - i \sin \frac{a}{2} l_+ \right). \end{aligned} \quad (85)$$

Here  $\lambda$  and  $a$  define the form of the triangle with the accuracy up to the similarity transition (which we will not consider, taking the length of the 6-vector  $\rho = const$ ). In addition, as it was demonstrated in Ref. 33, the variable  $a$  determines the relation between the momenta of inertia of the triangle.

In the following it will be convenient to return to  $\xi$  and  $\eta$  and connect them to the coordinates  $\vec{l}_+$  and  $\vec{l}_-$ :

$$\xi = \frac{1}{2}(ul_+ + u^*l_-), \quad \eta = -\frac{i}{2}(vl_+ - v^*l_-). \quad (86)$$

Here

$$\begin{aligned} u &= e^{-i\lambda/2} \cos \frac{a}{2} - ie^{i\lambda/2} \sin \frac{a}{2}, \quad u^* = e^{i\lambda/2} \cos \frac{a}{2} + ie^{-i\lambda/2} \sin \frac{a}{2}, \\ v &= e^{-i\lambda/2} \cos \frac{a}{2} + ie^{i\lambda/2} \sin \frac{a}{2}, \quad v^* = e^{i\lambda/2} \cos \frac{a}{2} - ie^{-i\lambda/2} \sin \frac{a}{2}. \end{aligned} \quad (87)$$

In these formulae the Euler angles which define the position of the triangle and the coordinates determining its deformation are explicitly separated. Note that these expressions are not just products of the functions of the Euler angles and the functions of the coordinates connected with the deformation, but sums of the products of these functions; this in fact reflects the relation of the deformation and the rotation.

Why the introduction of these coordinates makes sense becomes clear if we write the expressions for  $\xi$  and  $\eta$  in the form

$$\begin{aligned} \xi &= \left( \frac{1}{2} uu^* \right)^{1/2} \frac{1}{\sqrt{2}} \left( e^{i\psi_1} l_+ + e^{-i\psi_1} l_- \right), \\ \eta &= \left( \frac{1}{2} vv^* \right)^{1/2} \frac{1}{\sqrt{2}} \left( e^{i\psi_2} l_+ + e^{-i\psi_2} l_- \right). \end{aligned} \quad (88)$$

Here we have introduced the phases  $\psi_1$  and  $\psi$

$$u = \rho_1 e^{i\psi_1}, \quad v = \rho_2 e^{i\psi}, \quad (89)$$

from which it follows that

$$\frac{u}{u^*} = e^{2i\psi_1}, \quad \frac{v}{v^*} = e^{2i\psi}, \quad \psi_2 = \psi - \frac{\pi}{2}, \quad (90)$$

while

$$\Theta = \psi_1 - \psi_2 \quad (91)$$

is the angle between the vectors  $\xi$  and  $\eta$ :

$$\xi\eta = |\xi||\eta| \cos \Theta. \quad (92)$$

Making use of the relations

$$\xi^2 = \frac{1}{2} uu^*, \quad \eta^2 = \frac{1}{2} vv^*, \quad (93)$$

we can express the angle  $\Theta$  in terms of our variables:

$$\cos \Theta = \frac{\cos \lambda \sin a}{\sqrt{1 - \sin^2 \lambda \sin^2 a}}. \quad (94)$$

The irrational connection between the angle  $\Theta$  and the angles  $a$  and  $\lambda$  forces us to find different ways for the Fourier transformation.

Let us introduce the unit vectors  $n$  and  $m$  defined as

$$n = \frac{\xi}{|\xi|}, \quad m = \frac{\eta}{|\eta|}. \quad (95)$$

From (88) it is clear that

$$\begin{aligned} n &= \frac{1}{\sqrt{2}}(e^{i\psi_1}l_+ + e^{-i\psi_1}l_-), \\ m &= \frac{1}{\sqrt{2}}(e^{i\psi_2}l_+ - e^{-i\psi_2}l_-). \end{aligned} \quad (96)$$

It is reasonable to re-write these expressions in the form

$$\begin{aligned} n &= D_{01}^1\left(0, \frac{\pi}{2}, 0\right)(e^{i\psi_1}l_+ + e^{-i\psi_1}l_-), \\ m &= D_{01}^1\left(0, \frac{\pi}{2}, 0\right)(e^{i\psi_2}l_+ - e^{-i\psi_2}l_-) \end{aligned} \quad (97)$$

or, for the components of  $n$  and  $m$ ,

$$\begin{aligned} n^{(M_1)} &= \sum_{\mu_1} D_{0\mu_1}^1\left(0, \frac{\pi}{2}, \psi_1\right) D_{\mu_1 M_1}^1(l_+ l_-), \\ m^{(M_2)} &= \sum_{\mu_2} D_{0\mu_2}^1\left(0, -\frac{\pi}{2}, \psi_2\right) D_{\mu_2 M_2}^1(l_+ l_-). \end{aligned} \quad (98)$$

Let us remind that the components  $l_+$  and  $l_-$  can be expressed with the help of the Wigner  $D$ -functions

$$D_{mn}^1(\varphi_1, \theta, \varphi_2) = l_m^{(n)}. \quad (99)$$

In fact  $D(\varphi_1, \theta, \varphi_2) \equiv D(l_+ l_-)$ , and we have introduced this somewhat unusual notation in order to demonstrate that the Euler angles determine the position of the trihedron given by the unit vectors  $\vec{l}_1, \vec{l}_2, \vec{l}_3$ . The expression (98) describes the rotation which transforms the trihedron  $\vec{l}_1, \vec{l}_2, \vec{l}_3$  to one defined by the unit vector  $\vec{n}$  and two perpendicular vectors  $\vec{n}_1, \vec{n}_2$ . As a result of this rotation the vector  $\vec{n}$  turns out to be on the  $\vec{l}_1, \vec{l}_2$  plane. Similarly, formula (98) gives a description of turning to a trihedron defined by the unit vectors  $\vec{m}_1, \vec{m}_2, \vec{m}_3$ . Thus the expressions (98) can be considered as formulae for transforming first order Legendre polynomials (of vectors  $\vec{m}$  and  $\vec{n}$ ) which can be generalized to arbitrary polynomials.

Hence, for the polynomial build up from the unit vector  $\vec{n}$ , the following expression can be written:

$$D_{0M_1}^{j_1}(n) = \sum_{\mu_1=-j_1}^{j_1} D_{0\mu_1}^{j_1}\left(0, \frac{\pi}{2}, -\psi_1\right) D_{\mu_1 M_1}^{j_1}(\varphi_1, \theta, \varphi_2) \quad (100)$$

For the polynomial given by  $\vec{m}$  we have, respectively:

$$D_{0M_2}^{j_2}(m) = \sum_{\mu_2=-j_2}^{j_2} D_{0\mu_2}^{j_2}\left(0, \frac{\pi}{2}, -\varphi_2\right) D_{\mu_2 M_2}^{j_2}(\varphi_1, \theta, \varphi_2). \quad (101)$$

These formulae can be transformed in such a way that  $D_{0M_1}^{j_1}$  and  $D_{0M_2}^{j_0}$  turn out to be the functions of the same arguments:

$$\begin{aligned} D_{0M_1}^{j_1}(n) &= \sum_{\mu_1} \exp \left[ i\mu_1 \frac{(\psi_1 - \psi_2)}{2} \right] D_{0\mu_1}^{j_1} \left( 0, \frac{\pi}{2}, -\left( \frac{\psi_1 + \psi_2}{2} \right) \right) D_{\mu_1 M_1}^{j_1}(\varphi_1, \theta, \varphi_2), \\ D_{0M_2}^{j_2}(m) &= \sum_{\mu_2} \exp \left[ -i\mu_2 \frac{(\psi_1 - \psi_2)}{2} \right] D_{0\mu_2}^{j_2} \left( 0, \frac{\pi}{2}, -\left( \frac{\psi_1 + \psi_2}{2} \right) \right) D_{\mu_2 M_2}^{j_2}(\varphi_1, \theta, \varphi_2). \end{aligned} \quad (102)$$

For the three-body problem one more restriction has to be introduced. Since there is a definite reflection symmetry with respect to the moving plane of  $\vec{l}_+$  and  $\vec{l}_-$ ,  $\mu_1$  and  $\mu_2$  have to be either only even, or only odd. For example, in the sums (98) there can be only  $\mu_{1,2} = \pm 1$ .

We have introduced above quite a diversity of parameters and coordinates, and carried out a lot of transformations which, so far, may seem to be superfluous and somewhat artificial. In fact, as we will see, they simplify the calculations of the Fourier coefficients of the polynomials.

### 3.2. The case of $J = 0$

For states with a total momentum  $J = 0$  the solution can be easily obtained. From the equation for this case we got the solution in the form

$$D_{\nu/2, -\nu/2}^{K/4}(2\lambda, 2a, 0). \quad (103)$$

The order  $K/4$  of the  $D$ -function corresponds to the order  $K$  of the polynomial, since the latter is determined by the trigonometric functions of the argument  $a/2$ , and so each trigonometric function of  $2a$  increases the degree by four units.

The structure of (103) can be understood without considering the equation. Indeed, the polynomial at  $J = 0$  can not contain vectors  $\vec{l}_+$  and  $\vec{l}_-$ , and, hence, has to be the function of  $z^2$  and  $z^{*2}$ . We know that

$$\begin{aligned} z^2 &= i \sin a e^{-i\lambda}, \\ z^{*2} &= -i \sin a e^{i\lambda}. \end{aligned} \quad (104)$$

Having a look at the tables for  $D$ -functions it becomes clear that

$$\begin{aligned} z^2 &= D_{-1/2, 1/2}^{1/2}(2\lambda, 2a, 0), \\ z^{*2} &= -D_{1/2, -1/2}^{1/2}(2\lambda, 2a, 0), \end{aligned} \quad (105)$$

and that the sum of the lower indices is zero for each  $D$ -function. When constructing harmonic polynomials from functions (105), this feature, obviously, remains valid also for higher order  $D$ -functions (which is a reason for (103)). Consequently, a specific property of our problem is the absence of the diagonal elements

$$D_{1/2, 1/2}^{1/2}, \quad D_{-1/2, -1/2}^{1/2} \quad (106)$$

in the basis. The interesting task of the expansion of function (103) over the “tree”-functions, *i.e.* over functions with pair angular momenta arises; the state of the system at  $J = 0$  is described as a rotation of vectors  $\xi$  and  $\eta$  in opposite directions, with different momenta. This leads to a connection with the theory of Clebsch-Gordan coefficients,

$$\begin{pmatrix} K/4 & K/4 & j \\ \nu/2 & -\nu/2 & 0 \end{pmatrix}. \quad (107)$$

Let us note that the amplitudes of states with momenta  $j_1 = j_2 = j$  turn out to be proportional to the Wigner coefficient, the respective calculations are given below.

### 3.2.1. States with momenta $j_1 = j_2 = j$ and the Wigner coefficients

To obtain the contribution of the partial momenta in the  $J = 0$  state, we have to calculate the Fourier coefficient of the function

$$\Phi_0(\xi, \eta) = (\cos \Phi)^J (\sin \Phi)^j P_{K/2-j}^{(j+1/2, j+1/2)}(\cos 2\Phi) P_j(n, m). \quad (108)$$

As it was said already, at  $\xi^2 + \eta^2 = 1$  we suppose  $\cos^2 \Phi = \xi^2$  and  $\sin^2 \Phi = \eta^2$ , and re-write (108) in the form

$$\Phi_0(\xi, \eta) = (\xi^2)^{j/2} P_{K/2-j}^{(j+1/2, j+1/2)}(\xi^2 - \eta^2) P_j(n, m). \quad (109)$$

A state with zero angular momentum can be constructed from two partial momenta  $j_1$  and  $j_2$  which are equal to each other ( $j_1 = j_2 = j$ ). Since for such a state the quantum number  $\Omega$  plays no role, it differs from the states presented above only by the substitution of  $\nu$  by  $j$ . Hence, the function (109) is a superposition

$$\sum_{\nu} C(j, \nu) D_{\nu/2, -\nu/2}^{K/4}(2\lambda, 2a, 0). \quad (110)$$

The coefficient  $C(j, \nu)$  has to be calculated. The Fourier coefficient of (109) will be obtained having an additional condition,  $\vec{m}\vec{n} = 1$ ; this corresponds to  $a = \pi/2$  in (94). On the other hand,

$$\cos 2\Phi = \sin a \sin \lambda,$$

which, if  $a = \pi/2$ , gives

$$\cos 2\Phi = \sin \lambda, \quad \sin 2\Phi = \cos \lambda. \quad (111)$$

In order to be able to use the standard formulae, we take  $\sin \lambda = \cos \Lambda$ , and change from the Jacobi polynomial to the Gegenbauer polynomial:

$$P_{K/2-j}^{(j+1/2, j+1/2)}(\cos 2\Phi) = \frac{\Gamma(2j+2)\Gamma(K+3/2)}{\Gamma((K/2)+j+2)\Gamma(j+3/2)} C_{K/2-j}^{j+1}(\cos 2\Phi). \quad (112)$$

If so, (108) can be re-written in the form

$$(\cos \Phi)^j (\sin \Phi)^J = \frac{\Gamma(2j+2)\Gamma(K+3/2)}{\Gamma((K/2)+j+2)\Gamma(j+3/2)} C_{K/2-j}^{j+1}(\cos 2\Phi). \quad (113)$$

Let us make use now of the integral representation of the Gegenbauer polynomial:<sup>45</sup>

$$\begin{aligned} \frac{1}{2^j} (\sin \lambda)^j C_{K/2-j}^{j+1}(\cos \lambda) &= \frac{j^j}{2^{2j+1}} \frac{\Gamma(2+j+(K/2))}{(K/2)!\Gamma(j+1)} \\ &\times \int_0^\pi (\cos \Lambda - i \sin \Lambda \cos \vartheta)^{K/2} C_j^{1/2}(\cos \vartheta) \sin \vartheta d\vartheta. \end{aligned} \quad (114)$$

Taking into account  $C_j^{1/2}(\cos \vartheta) = P_j(\cos \vartheta)$ , we have

$$\begin{aligned} \Phi_0(\xi, \eta) &= \frac{i^j}{2^{2j+1}} \frac{\Gamma(2j+2)\Gamma(K+3/2)}{(K/2)!\Gamma(j+1)\Gamma(j+3/2)} \\ &\times \int_0^\pi (\cos \lambda - i \sin \Lambda \cos \vartheta)^{\frac{K}{2}} P_j(\cos \vartheta) \sin \vartheta d\vartheta. \end{aligned} \quad (115)$$

The expansion into a series can be easily obtained directly, if we first expand  $\cos \Lambda$  and  $\sin \Lambda$  in exponents and open the brackets. Extracting the term with the exponent  $e^{-i\nu\Lambda}$ , we get

$$\begin{aligned} \left(\frac{K}{4} - \frac{\nu}{2}\right) \int_0^\pi \left(\frac{1 - \cos \vartheta}{2}\right)^{\frac{K/2-\nu}{2}} \left(\frac{1 + \cos \vartheta}{2}\right)^{\frac{K/2+\nu}{2}} P_j(\cos \vartheta) e^{-i\nu\lambda} \sin \vartheta d\vartheta &= \\ = \left(\frac{K}{4} - \frac{\nu}{2}\right) \int_0^\pi \left(\sin \frac{\vartheta}{2}\right)^{K/2-\nu} \left(\cos \frac{\vartheta}{2}\right)^{K/2+\nu} P_j(\cos \vartheta) e^{-i\nu\Lambda} \sin \vartheta d\vartheta. \end{aligned} \quad (116)$$

This can be re-written as

$$i^{-K/2} \int_0^\pi P_{K/4, \nu/2}^{K/4}(\cos \vartheta) P_{-K/4, -\nu/2}^{K/4}(\cos \vartheta) P_{00}^j(\cos \vartheta) e^{-i\nu\Lambda} \sin \vartheta d\vartheta, \quad (117)$$

leading to the expression

$$\begin{aligned} \Phi_0(\xi, \eta) &= \sum_\nu \frac{i^{j-K/2}}{2^{2j} \sqrt{2j+1}} \frac{\Gamma(2j+2)\Gamma(K+3/2)}{\Gamma(j+1)\Gamma(j+3/2)} \times \\ &\times \left(\frac{K}{4}, \frac{\nu}{2}, -\frac{\nu}{2} \middle| j0\right) e^{-i\nu\lambda} \left[\left(\frac{K}{2} - j\right)! \left(\frac{K}{2} + j + 1\right)!\right]^{-1/2}. \end{aligned} \quad (118)$$

If, instead of (108), we begin with the orthonormal function

$$\Phi_0^{j+1/2, j+1/2}(\xi, \eta) = \left(N_{K/2-j}^{j+1/2, j+1/2}\right)^{-1/2} \Phi_0(\xi, \eta) \quad (119)$$

where

$$N_{K/2-j}^{j+1/2, j+1/2} = \frac{[i\Gamma(K/2 + 3/2)]^2}{2(K+2)\Gamma(K/2 - j + 1)\Gamma(K/2 + j + 2)}, \quad (120)$$

the calculation leads to the following expansion of the “tree”-functions into  $K$ -harmonics:

$$\begin{aligned} \Phi_0^{j+1/2, j+1/2}(\xi, \eta) = & - \sum_{\nu} \frac{i^j}{2^{2j-1/2}} \left( \frac{K+2}{2j+1} \right)^{1/2} \frac{\Gamma(2j+2)\Gamma(K+3/2)}{\Gamma(j+1)\Gamma(j+3/2)\Gamma((K+3)/2)} \\ & \times \left( \frac{K}{4}, \frac{\nu}{2}, \frac{K}{4}, -\frac{\nu}{2} \middle| j0 \right) D_{\nu/2, -\nu/2}^{K/4}(2\lambda, 2a, 0). \end{aligned} \quad (121)$$

Here we skipped the condition  $a = \pi/2$ . Obviously, the expansion of the  $K$ -harmonics into the “tree”-functions is determined by the same coefficients as the expansion (121).

### 3.3. The “tree”-functions

The solution of the Laplace equation on the five-dimensional sphere can be written in a coordinate system corresponding to the “tree” (see Fig. 1). In this system we

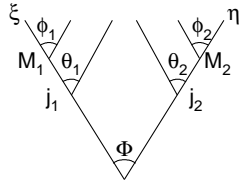


Fig. 1. “Tree”

presume

$$\xi = \cos \Phi n, \quad \eta = \sin \Phi m, \quad (122)$$

taking into account that  $\xi^2 + \eta^2 = 1$ . The eigenfunction related to this “tree” can be build up in the following way.<sup>42</sup> The junction  $n$  (where  $2n = K - j_1 - j_2$ ) corresponds to the function

$$(\cos \Phi)^{j_1} (\sin \Phi)^{j_2} P_n^{(j_1+1/2, j_2+1/2)}(\cos 2\Phi). \quad (123)$$

Due to (122)

$$\cos^2 \Phi = \xi^2, \quad \sin^2 \Phi = \eta^2, \quad (124)$$

and, hence, the expression (123) can be re-written in the form

$$\xi^{j_1} \eta^{j_2} P_{(K-j_1-j_2)/2}^{(j_1+1/2, j_2+1/2)}(\xi\eta), \quad (125)$$

where

$$P_{K-j_1-j_2/2}^{(j_1+1/2, j_2+1/2)}(\xi, \eta) = \sum_{m=0}^N \binom{n+j_1+1/2}{m} \binom{n+j_2+1/2}{n-m} (\xi^2)^m (\eta^2)^{n-m} (-1)^{n-m}. \quad (126)$$

The “branches”  $\xi$  and  $\eta$  characterized by  $j_1 M_1$  and  $j_2 M_2$  correspond, accordingly, to the functions

$$\begin{aligned} P_{j_2 M_2}(n) &= P_{j_1}^{M_1}(\vartheta_1) e^{-i M_1 \varphi_1}, \\ P_{j_2 M_2}(m) &= P_{j_2}^{M_2}(\vartheta_2) e^{-M_2 \varphi_2}. \end{aligned} \quad (127)$$

Multiplying the functions (126) and (127), we arrive at an eigenfunction which corresponds to the “tree” as a whole:

$$\Phi(\xi, \eta) = P_{J, M}(\xi, \eta) P_{K-j_1-j_2/2}^{(j_1+1/2, j_2+1/2)}(\xi \eta), \quad (128)$$

where

$$\begin{aligned} P_{J_1 M}(\xi, \eta) &= P_{j_1 M_1}(\xi) P_{j_2 M_2}(\eta), \\ P_{j_1 M_1}(\xi) &= \xi^{j_1} P_{j_1 M_1}(n), \quad P_{j_2 M_2}(\eta) = \eta^{j_2} P_{j_2 M_2}(m). \end{aligned} \quad (129)$$

Making use of the relations (90) and (93), we turn now to the variables  $u, v, u^*$  and  $v^*$  and re-write the expressions (129) in the form

$$\begin{aligned} P_{j_1 M_1}(\xi) &= i^{M_1} \left[ \frac{(j_1 + M_1)!}{(j_1 - M_1)!} \right]^{1/2} \\ &\times \sum_{\mu_1} \frac{1}{2^{j_1}} u^{(j_1+\mu_1)/2} (u^*)^{(j_1-\mu_1)/2} \Delta_{0\mu_1}^{(j_1)} D_{\mu_1 M_1}^{j_1}(\varphi_1, \theta, \varphi_2), \\ P_{j_2 M_2}(\eta) &= i^{M_2} \left[ \frac{(j_2 + M_2)!}{(j_2 - M_2)!} \right]^{1/2} \\ &\times \sum_{\mu_2} \frac{(-i)^{1/2}}{2^{-j_2}} v^{(j_2+\mu_2)/2} (v^*)^{(j_2-\mu_2)/2} \Delta_{0\mu_2}^{(j_2)} D_{\mu_2 M_2}^{j_2}(\varphi_1, \theta, \varphi_2). \end{aligned} \quad (130)$$

Here the notation

$$\Delta_{kl}^{(m)} = D_{kl}^m \left( 0, \frac{\pi}{2}, 0 \right) \quad (131)$$

is used.

Expanding the product  $D_{\mu_1 M_1}^{j_1}(\varphi_1, \theta, \varphi_2) D_{\mu_2 M_2}^{j_2}(\varphi_1, \theta, \varphi_2)$  over the functions  $D_{\mu M}^j(\varphi_1, \theta, \varphi_2)$  (where  $M = M_1 + M_2$ ,  $\mu = \mu_1 + \mu_2$ ) and extracting from the sum one term with a definite  $J$  (where  $|j_1 - j_2| \leq J \leq j_1 + j_2$ ), we obtain

$$\begin{aligned} P_{J, M}(\xi, \eta) &= i^M \left[ \frac{(j_1 + M_1)!(j_2 + M_2)!}{(j_1 - M_1)!(j_2 - M_2)!} \right]^{1/2} (j_1 j_2, 00 | J 0) (j_1 j_1, M_1 M_2 | J M) \\ &\times \sum_{\mu_1, \mu_2} \frac{(-i)^{\mu^2}}{2^{j_1+j_2}} (u^*)^{j_1-\mu_1/2} v^{(j_2+\mu_2)/2} u^{(j_1+\mu_1)/2} (v^*)^{(j_2-\mu_2)/2} \\ &\times (j_1 j_2 \mu_1 \mu_2 | J \mu)^2 \Delta_{0\mu}^{(J)} D_{\mu M}^J(\varphi_1, \theta, \varphi_2). \end{aligned} \quad (132)$$



With the help of (90) and (93), the expression (126) can be re-written in the form

$$P_{(K-j_1-j_2)/2}^{(j_1+1/2, j_2+1/2)}(\xi, \eta) = \sum_m \frac{1}{2^n} \binom{n+j_1+1/2}{m} \binom{n+j_2+1/2}{n-m} \times (-1)^{n-m} (uu^*)^m (vv^*)^{n-m}. \quad (133)$$

Inserting the formulae (132) and (133) into (128), we obtain for the eigenfunction  $\Phi(\xi, \eta)$  the following expression:

$$\Phi(\xi, \eta) = A'_{JM} \sum_m \sum_{\mu_1 \mu_2} \frac{(-i)^{\mu_2}}{2^{j_1+j_2+n}} (-1)^{n-m} (j_1 j_2, \mu_1 \mu_2 | J \mu)^2 \Delta_{0\mu}^{(J)} \binom{n+j_1+\frac{1}{2}}{m} \times \binom{n+j_2+\frac{1}{2}}{n-m} (u^*)^{\frac{j_1-\mu_1}{2}+m} v^{\frac{j_2+\mu_2}{2}+n-m} u^{\frac{j_1+\mu_1}{2}+n-m} D_{\mu M}^J(\varphi_1, \theta, \varphi_2), \quad (134)$$

where

$$A'_{JM} = (-1)^{M/2} \left[ \frac{(j_1+M_1)!(j_2+M_2)!}{(j_1-M)!(j_2-M_2)!} \right]^{1/2} (j_1 j_2, 00 | J 0) (j_1 j_2, M_1 M_2 | J M). \quad (135)$$

As it was already mentioned, all the further calculations are necessary in order to obtain the function  $\Phi(\xi, \eta)$  in a form for which the Fourier transformation becomes relatively simple. Let us use the formulae given in Ref. 46:

$$\begin{aligned} & \frac{1}{\sqrt{(j-k)!(j+k)!}} \left( \cos \frac{\alpha}{2} e^{i\gamma/2} + i \sin \frac{\alpha}{2} e^{-i\gamma/2} \right)^{j-k} \times \\ & \times \left( i \sin \frac{\alpha}{2} e^{i\gamma/2} \cos \frac{\alpha}{2} e^{-i\gamma/2} \right)^{j+k} = \sum_{l=-j}^j \frac{P_{lk}^j(\cos \alpha)}{\sqrt{(j-l)!(j+l)!}} e^{-il\gamma}, \\ & \frac{1}{\sqrt{(j-k)!(j+k)!}} \left( \cos \frac{\alpha}{2} e^{-i\gamma/2} - i \sin \frac{\alpha}{2} e^{i\gamma/2} \right)^{j-k} \times \\ & \times \left( -i \sin \frac{\alpha}{2} e^{-i\gamma/2} + \cos \frac{\alpha}{2} e^{i\gamma/2} \right)^{j+k} = \sum_{l=-j}^j \frac{\bar{P}_{lk}^j(\cos \alpha)}{\sqrt{(j-l)!(j+l)!}} e^{il\gamma}. \quad (136) \end{aligned}$$

Comparing these expressions with the formulae (87) for  $u, u^*, v$  and  $v^*$  and remembering that

$$\bar{P}_{lk}^j(\cos \alpha) = (-1)^{l-k} P_{-l-k}^j(\cos \alpha), \quad (137)$$

we can write

$$\begin{aligned}
 (u^*)^{\frac{j_1-\mu_1}{2}+m} v^{\frac{j_2+\mu_2}{2}+n-m} &= \left[ \left( \frac{j_1-\mu_1}{2} + m \right)! \left( \frac{j_2+\mu_2}{2} + n - m \right)! \right]^{1/2} \\
 &\times \sum_{\nu_1=-(K-\delta)/4}^{(K-\delta)/4} P_{\nu_1, W+\mu/4}^{(K-\delta)/4}(\cos a) e^{-i\nu_1 \lambda} \left[ \left( \frac{K-\delta}{4} - \nu_1 \right)! \left( \frac{K-\delta}{4} + \nu_1 \right)! \right]^{-1/2}, \\
 u^{\frac{j_1+\mu_1}{2}+m} (v^*)^{\frac{j_2-\mu_2}{2}+n-m} &= \left[ \left( \frac{j_1+\mu_1}{2} + m \right)! \left( \frac{j_2-\mu_2}{2} + n - m \right)! \right]^{1/2} \\
 &\times \sum_{\nu_2=-(K+\delta)/4}^{(K+\delta)/4} P_{\nu_2, -W+\mu/4}^{(K+\delta)/4}(\cos a) (-1)^{\nu_2-W+\mu/4} e^{i\nu_2 \lambda} \\
 &\times \left[ \left( \frac{K+\delta}{4} - \nu_2 \right)! \left( \frac{K+\delta}{4} + \nu_2 \right)! \right]^{-1/2}, \tag{138}
 \end{aligned}$$

where  $\delta = \mu_1 - \mu_2$ ,  $W = \frac{1}{4}(j_2 - j_1) + \frac{1}{2}n - m$ .

Then, expanding  $P_{\nu_1, W+\mu/4}^{\frac{1}{4}(K-\delta)}(\cos a)$  and  $P_{-\nu_2, -W+\frac{1}{4}\mu}^{(K+\frac{1}{4}\delta)}(\cos a)$  over the functions  $P_{\nu, \mu/2}^{K/2-\kappa}(\cos a)$  (where  $\nu = \nu_1 - \nu_2$ ), the eigenfunction  $\Phi(\xi, \eta)$  can be re-written in the form

$$\begin{aligned}
 \Phi(\xi, \eta) &= A'_{JM} \sum_{m, \mu, \delta, \nu, \varepsilon, \kappa} \frac{1}{2^{j_1+j_2+n}} (-1)^{\frac{K-\delta}{4} + \frac{\varepsilon-\nu}{2} + \frac{\mu}{2} - \frac{j_2}{2}} \left( j_1, j_2, \frac{\mu+\delta}{2}, \frac{\mu-\delta}{2} \middle| J, \mu \right)^2 \\
 &\times \left( \frac{K-\delta}{4}, \frac{K+\delta}{4}, \frac{\nu+\varepsilon}{2}, \frac{\nu-\varepsilon}{2} \middle| \frac{K}{2} - \kappa, \nu \right) \\
 &\times \left( \frac{K-\delta}{4}, \frac{K+\delta}{4}, W + \frac{\mu}{4}, -W + \frac{\mu}{4} \middle| \frac{K}{2} - \kappa, \frac{\mu}{2} \right) \\
 &\times \Delta_{0, \mu}^{(J)} \binom{n+j_1+1/2}{m} \binom{n+j_2+1/2}{n-m} \\
 &\times \left[ \left( \frac{j_1}{2} - \frac{\mu+\delta}{4} + m \right)! \left( \frac{j_1}{2} + \frac{\mu+\delta}{4} + m \right)! \right. \\
 &\quad \times \left. \left( \frac{j_2}{2} + \frac{\mu-\delta}{4} + n - m \right)! \left( \frac{j_2}{2} - \frac{\mu-\delta}{4} + n - m \right)! \right]^{\frac{1}{2}} \\
 &\times \left[ \left( \frac{K-\delta}{4} - \frac{\nu+\varepsilon}{2} \right)! \left( \frac{K-\delta}{4} + \frac{\nu+\varepsilon}{2} \right)! \right. \\
 &\quad \times \left. \left( \frac{K+\delta}{4} - \frac{\nu-\varepsilon}{2} \right)! \left( \frac{K+\delta}{4} + \frac{\nu-\varepsilon}{2} \right)! \right]^{-\frac{1}{2}} \\
 &\times P_{\nu, \mu/2}^{K/2-\kappa}(\cos a) D_{\mu, M}^J(\cos a) D_{\mu, M}^J(\varphi_1, \theta, \varphi_2) e^{-i\nu \lambda}. \tag{139}
 \end{aligned}$$

Making use of several relations for the Clebsch-Gordan coefficients, we can change

(139) so that, introducing

$$P_k^{(\alpha, \beta)}(0) = \frac{1}{2^k} \sum_{m=0}^k \binom{k+\alpha}{m} \binom{k+\beta}{k-m} (-1)^{k-m}, \quad (140)$$

the sum over  $\epsilon = \nu_1 + \nu_2$  can be easily taken.

In the following we shall consider only one definite term of the sum over  $\nu$ . After some rather cumbersome algebraic transformations we obtain the final result

$$\begin{aligned} \Phi(\xi, \eta) = & A_{JM} \sum_m \sum_{\mu\delta} \sum_{\kappa} \left( j_1, j_2, \frac{\mu+\delta}{2}, \frac{\mu-\delta}{2} \middle| j\mu \right)^2 \\ & \times \left( \frac{K-\delta}{4}, \frac{K+\delta}{4}, W + \frac{\mu}{4}, -W + \frac{\mu}{4} \middle| \frac{K}{2} - \kappa, \frac{\mu}{2} \right) \\ & \times \left( \frac{j_1}{2} + m, \frac{j_2}{2} + n - m, \frac{\mu+\delta}{4}, \frac{\mu-\delta}{4} \middle| \frac{K}{2}, \frac{\mu}{2} \right)^{-1} \\ & \times \frac{(-1)^{\frac{K+\mu-\delta}{4} - \frac{\nu}{4} + \kappa}}{2^{K/4}} \frac{\Delta_{0\mu}^{(J)} \Delta_{\delta/2, \nu}^{(K/2-\kappa)}}{\Delta_{K/2, \mu/2}^{K/2}} \\ & \times \sqrt{\frac{(K-2\kappa+1)}{(K+\kappa+1)! \kappa!}} \binom{n+j_1+1/2}{m} \binom{n+j_2+1/2}{n-m} \\ & \times \sqrt{(j_1+2m)!(j_2+2n-2m)!} D_{\nu, \mu/2}^{K/2-\kappa}(\lambda, a, 0) D_{\mu, M}^J(\varphi_1, \theta, \varphi_2), \quad (141) \end{aligned}$$

where  $A_{JM} = [(-1)^{-j_2/2}/2^{(j_1+j_2)/2}] A'_{JM}$ .

Thus we obtained an expression for the general solution of the problem in the form (79). (In our case the notations  $M' = \mu/2, \Lambda/2 - \kappa$  are used.) From the structure of the coefficients at  $D_{\nu, \mu/2}^{K/2-\kappa}(\lambda, a, 0) D_{\mu, M}^J(\varphi_1, \theta, \varphi_2)$  it becomes clear why in Ref. 33 the explicit form of  $a_\nu(\Lambda, M')$  could not be determined.

### 3.4. A different way of obtaining the eigenfunction $\phi(\xi, \eta)$

Calculating the explicit form of the eigenfunction we have noticed that, instead of the product of two  $D$ -functions, there exists a probably more convenient form of  $\Phi(\xi, \eta)$ .

Let us start with the formula (134). Taking the explicit expressions of  $u, v, u^*$

and  $v^*$  (87), we expand them into a series over the degrees of  $\sin(a/2)$  and  $\cos(a/2)$ :

$$\begin{aligned}
 u^A &= \sum_{s=-A/2}^{A/2} \left( \frac{A}{A+s} \right) \left( \cos \frac{a}{2} \right)^{(A+s)/2} (-i)^{(A+s)/2} e^{i\lambda s/2} \\
 v^B &= \sum_{t=-B/2}^{B/2} \left( \frac{B}{B+t} \right) \left( \cos \frac{a}{2} \right)^{(B-t)/2} \left( \sin \frac{a}{2} \right)^{(B+t)/2} (i)^{(B+t)/2} e^{s\lambda t/2}, \\
 u^{*C} &= \sum_{u=-C/2}^{C/2} \left( \frac{C}{C+u} \right) \left( \cos \frac{a}{2} \right)^{(C+u)/2} \left( \sin \frac{a}{2} \right)^{(C-u)/2} e^{i\lambda u/2}, \\
 v^{*D} &= \sum_{v=-D/2}^{D/2} \left( \frac{D}{D+v} \right) \left( \cos \frac{a}{2} \right)^{(D+v)/2} \left( \sin \frac{a}{2} \right)^{(D-v)/2} (-i)^{(D-v)/2} e^{i\lambda v/2}. \quad (142)
 \end{aligned}$$

With the help of these relations we can write

$$\begin{aligned}
 &(u^*)^{\frac{j_1-\mu_1}{2}+m} v^{\frac{j_2+\mu_2}{2}+n-m} u^{\frac{j_1+\mu_1}{2}+m} (v^*)^{\frac{j_2-\mu_2}{2}+n-m} = \\
 &= \sum_{\substack{s,t,u,v \\ s+t+u+v=-2\nu}} \left( \frac{\frac{j_1+\mu_1}{2}+m}{\frac{(j_1+\mu_1)/2+m+s}{2}} \right) \left( \frac{\frac{j_2+\mu_2}{2}+n-m}{\frac{(j_2+\mu_2)/2+n-m+t}{2}} \right) \\
 &\times \left( \cos \frac{a}{2} \right)^{\frac{E-s-t+u+v}{2}} \left( \sin \frac{a}{2} \right)^{\frac{K+s+t-u-v}{2}} (i)^{\frac{-\delta-s+t-u+v}{2}} e^{i[(s+t+u+v)/2]\lambda}. \quad (143)
 \end{aligned}$$

Making use of (140), let us introduce  $P_k^{(\alpha,\beta)}(0)$ . If so, the two sums can be easily calculated, and we have

$$\begin{aligned}
 \Phi(\xi, \eta) &= A'_{JM} \sum_m \sum_{\mu\delta} \frac{(-i)^{(\mu-\delta)/2}}{2^{j_1+j_2+n}} (-1)^{n-m} \left( j_1, j_2, \frac{\mu+\delta}{2}, \frac{\mu-\delta}{2} \middle| J\mu \right)^2 \\
 &\times \Delta_{0\mu}^{(J)} \binom{n+j_1+1/2}{n-m} \binom{n+j_2+1/2}{n-m} \sum_{\substack{t,u \\ s+t+u+v=-2\nu}} (i)^{-\frac{K}{2}-\frac{\delta}{2}} 2^{\frac{K}{2}} e^{-i\nu\lambda} \\
 &\times \left[ \left( \frac{j_1}{2} + \frac{\mu+\delta}{4} + m \right)! \left( \frac{j_1}{2} - \frac{\mu+\delta}{4} + m \right)! \left( \frac{j_2}{2} + \frac{\mu-\delta}{4} + n-m \right)! \right. \\
 &\times \left. \left( \frac{j_2}{2} - \frac{\mu-\delta}{4} + n-m \right)! \right]^{1/2} \left[ \left( \frac{K}{4} + \frac{\mu}{4} + \frac{s+t}{2} \right)! \left( \frac{K}{4} + \frac{\mu}{4} - \frac{s+t}{2} \right)! \right. \\
 &\times \left. \left( \frac{K}{4} - \frac{\mu}{4} + \frac{u+v}{2} \right)! \left( \frac{K}{4} - \frac{\mu}{4} - \frac{u+v}{2} \right)! \right]^{-1/2} \\
 &\times \Delta_{-\frac{s+t}{2}, W-\frac{\delta}{4}}^{\left(\frac{K}{4}+\frac{\mu}{4}\right)} \Delta_{-\frac{u+v}{2}, -W-\frac{\delta}{4}}^{\left(\frac{K}{4}-\frac{\mu}{4}\right)} \\
 &\times \left( \cos \frac{a}{2} \right)^{\frac{K}{2}-\frac{(s+t)-(u+v)}{2}} \left( \sin \frac{a}{2} \right)^{\frac{K}{2}+\frac{(s+t)-(u+v)}{2}} D_{\mu M}^J(\varphi_1, \theta, \varphi_2). \quad (144)
 \end{aligned}$$

This expression can be re-written in the form

$$\begin{aligned} \Phi(\xi, \eta) = \sum_{\mu, \delta, \sigma, W} N(K, \nu, j_1, j_2 | \mu, \delta, \sigma, W) \times \\ \times (\cos a + 1)^{\frac{K}{4} - \frac{\sigma}{4}} (\cos a - 1)^{\frac{K}{4} + \frac{\sigma}{4}} D_{\mu M}^J(\varphi_1, \theta \varphi_2) e^{-i\nu\lambda}, \end{aligned} \quad (145)$$

where

$$\begin{aligned} N(K, \nu, j_1, j_2 | \mu, \delta, \sigma, W) = A_{JM} (-1)^{n-m} \left( j_1, j_2, \frac{\mu + \delta}{2}, \frac{\mu - \delta}{2} \middle| J_\mu \right)^2 \times \\ \times \Delta_{0\mu}^{(J)} \binom{n + j_1 + 1/2}{m} \binom{n + j_2 + 1/2}{n - m} \tilde{\Delta}_{\frac{\mu}{2} - \frac{\sigma}{4}, W - \frac{\sigma}{4}}^{(\frac{K}{4} + \frac{\mu}{4})} \tilde{\Delta}_{\frac{\mu}{2} + \frac{\sigma}{4}, -W - \frac{\sigma}{4}}^{(\frac{K}{4} - \frac{\mu}{4})} \end{aligned} \quad (146)$$

and we have introduced the notations  $\sigma = s + t - (u + v)$  and

$$\sqrt{\frac{(l-n)!(l+n)!}{(l-m)!(l+m)!}} \Delta_{mn}^{(l)} = \tilde{\Delta}_{mn}^{(l)}. \quad (147)$$

$A_{JM}$  is a normalization factor which we will not calculate. The summation limits in (145) are given by

$$-K \leq \sigma \leq K, \quad -\frac{K-2j_2}{4} \leq W \leq \frac{K-2j_1}{4}. \quad (148)$$

The limits in  $\mu$  and  $\delta$  at a definite  $\sigma$  can be obtained when the factorials in the denominator of  $\tilde{\Delta}$  turn into zero.

### 3.5. Mini-conclusion

The construction of a basis for three free particles which realizes the representation of the group of rotation in the three-dimensional space and of the permutation group, turned out to be unexpectedly difficult. The set of equations determining the eigenfunctions of the problem appeared to be very complicated, and its solution could be found only by an unconventional method. We succeeded in building functions which satisfied only four of the five equations. Hence, the final solution (including the quantum number  $\Omega$ ) has to be calculated by inserting the linear combination of the solutions with different  $(j_1 j_2)$  into the equation for  $\Omega$ , obtained in Ref. 31. Let us underline, however, that the last, fifth equation can be easily solved in every concrete case. In order to find the coefficients it is sufficient to compare the higher orders of  $\cos(a/2)$  in the polynomials, this was done in Ref. 43.

Still, we think that the orthogonalization of the polynomials can be carried out in a more efficient way, may be even without the operator  $\Omega$ .

The wave functions which were obtained here allow to consider another problem, namely: how the rotational spectrum of the system appears. To do this, the characteristics of the superposition over the quantum numbers  $K$  and  $\Omega$  have to be investigated. It would be also interesting to find out whether our method can be applied to the motion of a heavy top and especially to the case of the Kowalewski

top<sup>47</sup> (for recent discussions see for example<sup>48, 49</sup> and references therein). A possible application of the presented technique is the classification of the Dalitz plots<sup>33</sup> and the calculation of the matrix elements of pair interactions.

#### 4. A Complete Set of Functions in the Quantum Mechanical Three-Body Problem

A complete set of basis functions for the quantum mechanical three body problem is here chosen in the form of hyperspherical functions. These functions are characterized by quantum numbers corresponding to the chain  $O(6) \supset SU(3) \supset O(3)$ . Equations are derived to obtain the basis functions  $m$  in an explicit form.

Elementary processes involving three interacting particles exhibit an extremely complicated structure. It is therefore important to have at least a complete understanding of systems consisting of non-interacting particles.

In any classification of multiparticle states it is important to diagonalize those variables which are known to be constants of motion from general invariance principles, one usually takes the total energy and the angular momentum. We can deal with the case of equal masses because of the evident changeover:

$$\frac{1}{\sqrt{m_i}} \frac{\partial}{\partial x_i} = \frac{\partial}{\partial X_i}. \quad (149)$$

Our aim here is to present a complete set of orthonormal functions, corresponding to three free particles. Doing so, we introduce hyperspherical functions, *i.e.* functions, which are defined on the five-dimensional sphere, and are eigenfunctions of the angular part of the six-dimensional Laplacian. They have to describe states with given angular momenta and definite permutation symmetry properties. This choice of functions is due to the invariance of the free Hamiltonian under the  $O(6)$  or the  $SU(3)$  group.

The classification of the three-particle states based on these groups according to the chain  $O(6) \supset SU(3) \supset O(3)$  gives four quantum numbers, namely:  $K$  – the six-dimensional momentum, corresponding to the eigenvalue of the six-dimensional Laplacian;  $J$  – the angular momentum and its projection  $M$ , and a number  $\nu$ , which characterizes the permutation symmetry. On the other hand the motion of a system of  $n$  particles in a given energy and momentum state can be defined by  $3n - 4$  parameters, and requires for its quantum-mechanical description  $3n - 4$  quantum numbers, *i.e.* five in the case of three particles. That means that the states labeled according to the chain above might be degenerate; this degeneracy can be eliminated either by the straightforward orthogonalization of the functions, or with the help of a hermitian operator  $\hat{\Omega}$ , which we take from the group  $O(6)$ , and which commutes with the  $O(3)$  generators. The eigenvalue of this operator is the fifth – missing – quantum number  $\Omega$ . Unfortunately, since  $\hat{\Omega}$  is a cubic operator, it leads to rather complicated eigenvalue equations.

#### 4.1. Casimir operators and eigenfunctions

Let us re-calculate the operators, the eigenvalues of which we are trying to find. They are

$$\Delta = |A_{ik}|^2, \quad A_{ik} = iz_i \frac{\partial}{\partial z_k} - iz_k^* \frac{\partial}{\partial z_i^*} \quad (150)$$

the  $SU(3)$  generators;

$$J_{ik} = \frac{1}{2}(A_{ik} - A_{ki}) = \frac{1}{2} \left( iz_i \frac{\partial}{\partial z_k} - iz_k \frac{\partial}{\partial z_i} + iz_i^* \frac{\partial}{\partial z_k^*} - iz_k^* \frac{\partial}{\partial z_i^*} \right), \quad (151)$$

Here  $\Delta$  is the Laplace operator on the five-dimensional sphere;  $A_{ik}$  are the generators of the three-dimensional rotation group. The scalar operator reads

$$N = \frac{1}{2} \sum_k \left( z_k \frac{\partial}{\partial z_k} - z_k^* \frac{\partial}{\partial z_k^*} \right) = \frac{1}{2i} \text{Sp } A, \quad (152)$$

the eigenvalue of which is  $\nu$ . Finally, the operator

$$\hat{\Omega} = \sum_{i,k,l} J_{ik} B_{kl} J_{li},$$

$$B_{ik} = \frac{1}{2}(A_{ik} + A_{ki}) = \frac{1}{2} \left( iz_i \frac{\partial}{\partial z_k} + iz_k \frac{\partial}{\partial z_i} - iz_i^* \frac{\partial}{\partial z_k^*} - iz_k^* \frac{\partial}{\partial z_i^*} \right) \quad (153)$$

is the generator of the group of deformations of the triangle.

The explicit expressions for these five commuting operators are the following. Using

$$ds^2 = |dz|^2 = g_{ik} q^i q^k = \varrho^2 \left[ \frac{1}{4} da^2 + \frac{1}{4} d\lambda^2 + \frac{1}{2} d\Omega_1^2 + \frac{1}{2} d\Omega_2^2 + \right. \\ \left. + d\Omega_3^2 - \sin a d\Omega_1 d\Omega_2 - \cos a d\Omega_3 d\lambda \right] + d\varrho^2, \quad (154)$$

where  $d\Omega_i$  are infinitesimal rotations about the moving axes, we obtain the Laplacian:

$$\Delta = g^{-1/2} \frac{\partial}{\partial q^i} g^{ik} g^{1/2} \frac{\partial}{\partial q^k} \\ = \left\{ \frac{\partial}{\partial a^2} + 2 \text{ctg} 2a \frac{\partial}{\partial a} + \frac{1}{\sin^2 a} \left( \frac{\partial^2}{\partial \lambda^2} + \cos a \frac{\partial^2}{\partial \lambda \partial \Omega_3} + \frac{1}{4} \frac{\partial^2}{\partial \Omega_3^2} \right) + \right. \\ \left. + \frac{1}{2 \cos^2 a} \left[ \frac{\partial^2}{\partial \Omega_1^2} + \sin a \left( \frac{\partial^2}{\partial \Omega_1 \partial \Omega_2} + \frac{\partial^2}{\partial \Omega_2 \partial \Omega_1} \right) + \frac{\partial^2}{\partial \Omega_2^2} \right] \right\}. \quad (155)$$

The explicit form of  $N$  is  $N = i(\partial/\partial \lambda)$ . If a harmonic function of  $\Phi$  is an eigenfunction of  $\Delta$ , it has to fulfil

$$\Delta \Phi = -K(K+4)\Phi, \quad N\Phi = \nu\Phi. \quad (156)$$

Let us note here that if a harmonic function belongs to the  $SU(3)$  representation  $(p, q)$ , then  $K = p + q$ ,  $\nu = 1/2(p - q)$ .

The operator  $J_{ik}$  has the form

$$J_{ik} = -\frac{i}{2} \varepsilon_{ikl} \left[ l_1^{(1)} \frac{\partial}{\partial \Omega_1} + l_2^{(1)} \frac{\partial}{\partial \Omega_2} + l^{(1)} \frac{\partial}{\partial \Omega_3} \right]. \quad (157)$$

We obtain

$$\begin{aligned} \hat{\Omega} = & -\frac{1}{4} \left\{ 2^{1/2} \left( \frac{\partial^2}{\partial \Omega_+^2} H_+ + \frac{\partial^2}{\partial \Omega_-^2} H_- \right) + \frac{\partial^2}{\partial \Omega_3^2} \frac{\partial}{\partial \lambda} + \Delta \ominus \frac{\partial}{\partial \lambda} - \right. \\ & - \frac{1}{\cos a} \left( \Delta \ominus - \frac{\partial^2}{\partial \Omega_3^2} + \frac{1}{2} \right) \frac{\partial}{\partial \Omega_3} + \\ & \left. + \operatorname{tg} a \left[ 1 \left( \frac{\partial^2}{\partial \Omega_+^2} - \frac{\partial^2}{\partial \Omega_-^2} \right) \frac{\partial}{\partial \Omega_3} - \frac{3}{2} \left( \frac{\partial^2}{\partial \Omega_+^2} + \frac{\partial^2}{\partial \Omega_-^2} \right) \right] \right\}, \end{aligned} \quad (158)$$

where

$$\begin{aligned} H_{\pm} &= 2^{-1/2} \left[ \frac{\partial}{\partial a} \pm 1 \frac{1}{\sin a} \frac{\partial}{\partial \lambda} \pm \frac{i}{2} \operatorname{ctg} a \frac{\partial}{\partial \Omega_3} \right], \\ \frac{\partial}{\partial \Omega_{\pm}} &= 2^{-1/2} \left( \frac{\partial}{\partial \Omega_1} \pm i \frac{\partial}{\partial \Omega_2} \right). \end{aligned} \quad (159)$$

Before writing the eigenfunctions of these five operators, we have to make a few remarks. One can show that for  $K < 4$  all states are simple; in the interval  $4 \leq K < 8$  doubly degenerated states show up; as  $K$  is growing, the number of degenerated states grows too, and at the value  $K = 4n$  an  $n$ -fold degeneration appears. Besides, states with  $J = 0$  and  $J = 0$  values are not degenerated. Consequently, for practical purposes it is enough to deal with four quantum numbers.

Let us look for the harmonic functions  $\Phi$  which satisfy the eigenvalue equations of the Laplace operator and the operator  $N$  with eigenvalues  $K(K+4)$  and  $\nu$ , respectively. The general form is the following:

$$\Phi'_{M\nu} = \sum_{\nu} \sum_{\mu} a_{\nu}(\kappa, \mu) D_{\nu(\mu/2)}^{(K/2)-\nu}(\lambda, a, 0) D_{\mu, M}^J(\varphi_1 \ominus \varphi_2). \quad (160)$$

It is easy to understand the meaning of this solution. One can consider the second  $D$ -function – which is the eigenfunction of  $J^2$  and  $J_3$  – as an eigenfunction of a rotating rigid top with the projection of the angular momentum on the moving axis equal to  $\mu$ . This projection is not conserved in our case, that is why we have to sum over different values of  $\mu$ . That is just the point where we need an additional operator to orthogonalize the obtained functions. The coefficients  $a_{\nu}(\kappa, \mu)$  have to be defined from the equations

$$\Delta \Phi'_{M\nu} = -K(K+4) \Phi'_{M\nu}, \quad \hat{\Omega} \Phi'_{M\nu} = \Omega \Phi'_{M\nu}. \quad (161)$$



These equations are unfortunately somewhat complicated:

$$\begin{aligned}
 & \sum_{\kappa, \mu} \left\{ \left[ a_{\nu}(\kappa, \mu - 2) \frac{1}{2} \sqrt{\left(\frac{K}{2} - \frac{\mu}{2} - \kappa + 1\right) \left(\frac{K}{2} + \frac{\mu}{2} - \kappa\right)} \times \right. \right. \\
 & \times \sqrt{(J - \mu + 2)(J - \mu + 1)(J + \mu - 1)(J + \mu)} + a_{\nu}(\kappa, \mu + 2) \frac{1}{2} \times \\
 & \times \sqrt{\left(\frac{K}{2} + \frac{\mu}{2} - \kappa + 1\right) \left(\frac{K}{2} - \frac{\mu}{2} - \kappa\right)} \sqrt{(J + \mu + 2)(J + \mu + 1)(J - \mu - 1)(J - \mu)} + \\
 & + a_{\nu}(\kappa, \mu) (\nu \mu^2 + J(J + 1) \nu - 4i\Omega) \left. \right] D_{\nu(\mu/2)}^{K/2-\nu}(\lambda, a, 0) D_{\mu M}^J(\varphi_1 \ominus \varphi_2) + a_{\nu}(\kappa, \mu) \times \\
 & \times \left[ \frac{\mu}{\cos a} \left( J(J + 1) - \mu^2 + \frac{1}{2} \right) D_{\nu(\mu/2)}^{K/2-\kappa}(\lambda, a, 0) D_{\mu, M}^J(\varphi_1 \ominus \varphi_2) + i \operatorname{tg} a \times \right. \\
 & \times \left( \left( \frac{\mu}{2} + \frac{3}{4} \right) \sqrt{(J - \mu)(J + \mu + 1)(J - \mu - 1)(J + \mu + 2)} D_{\nu, \mu/2}^{K/2-\kappa}(\lambda, a, 0) \times \right. \\
 & \times D_{\mu+2M}^J(\varphi_1 \ominus \varphi_2) - \left( \frac{\mu}{2} - \frac{3}{4} \right) \sqrt{(J + \mu)(J - \mu + 1)(J + \mu - 1)(J - \mu + 2)} \times \\
 & \left. \left. \times D_{\nu(\mu/2)}^{K/2-\nu}(\lambda, a, 0) D_{\mu-2, M}^J(\varphi_1 \ominus \varphi_2) \right) \right] \Bigg\} = 0, \tag{162}
 \end{aligned}$$

and

$$\begin{aligned}
 & \sum_{\kappa, \mu} \left\{ a_{\nu}(\kappa, \mu) \left[ - \left( \frac{K}{2} - \kappa \right) \left( \frac{K}{2} - \kappa + 1 \right) + \frac{1}{4} K(K + 4) - \frac{\mu}{2} + \frac{\nu}{\cos a} - \right. \right. \\
 & - \frac{1}{2 \cos^2 a} J(J + 1) + \frac{\mu^2}{2 \cos^2 a} \left. \right] D_{\nu(\mu/2)}^{K/2-\kappa}(\lambda, a, 0) D_{\mu, M}^J(\varphi_1 \ominus \varphi_2) - \\
 & - a_{\nu}(\kappa, \mu - 2) i \operatorname{tg} a \sqrt{\left(\frac{K}{2} - \frac{\mu}{2} - \kappa + 1\right) \left(\frac{K}{2} + \frac{\mu}{2} - \kappa\right)} D_{\nu(\mu/2)}^{K/2-\nu}(\lambda, a, 0) \times \\
 & \times D_{\mu-2, M}^J(\varphi_1 \ominus \varphi_2) + a_{\nu}(\kappa, \mu) \left[ - \frac{i}{4} \frac{\sin a}{\cos^2 a} \times \right. \\
 & \times \sqrt{(J - \mu)(J + \mu + 1)(J - \mu - 1)(J + \mu + 2)} \times \\
 & \times D_{\nu(\mu/2)}^{K/2-\kappa}(\lambda, a, 0) D_{\mu+2M}^J(\varphi_1 \ominus \varphi_2) + \frac{i}{4} \frac{\sin a}{\cos^2 a} \times \\
 & \times \sqrt{(J + \mu)(J - \mu + 1)(J + \mu - 1)(J - \mu + 2)} \\
 & \left. \left. \times D_{\nu(M/2)}^{K/2-\kappa}(\lambda, a, 0) D_{\mu-2, M}^J(\varphi_1 \ominus \varphi_2) \right] \right\} = 0, \tag{163}
 \end{aligned}$$

and, although it is quite easy to solve this set of equations for every particular case, we have not been able so far to obtain a general solution.

#### 4.2. Another way of constructing a set of eigenfunctions

There is another way to find this complete set of functions. In fact, the problem becomes complicated because of the requirement of definite permutation symmetry

properties. Without them it would be simple to construct the wanted functions with the help of the graphical method of the so-called “tree-functions”,<sup>42</sup> which was proposed by Vilenkin and Smorodinsky. We have to modify these functions, *i.e* we have to find a transformation from the complete set of “tree-functions” to the  $K$  harmonics. ( $K$  harmonics are hyperspherical functions possessing definite permutation symmetry properties). Thus we first construct the “tree-functions”, which are characterized by quantum numbers

$$K, j_1, M_1, j_2, M_2$$

( $j_1, M_1$  and  $j_2, M_2$  are angular momenta and their projections conjugated to  $\vec{\xi}$  and  $\vec{\eta}$ ). We have to transform these functions to a set of  $K$  harmonics which is described by the quantum numbers  $K, J, M, \nu, (j_1 j_2)$ . In order to do this it is necessary to carry out a simple Fourier transform. To be correct,  $(j_1 j_2)$  is not a real quantum number in the sense that functions corresponding to different pairs  $(j_1 j_2)$  do not form an orthogonal set, but this notation demonstrates where we get these functions from. Their explicit expression is the following:

$$\begin{aligned} \Phi_{JM\nu}^{j_1 j_2}(\vec{\xi}, \vec{\eta}) = & A_{JM} \sum_m \sum_{n, \delta} \sum_{\kappa} \left( j_1, \frac{\mu + \delta}{2}; j_2, \frac{\mu - \delta}{2} \middle| J; \mu \right)^2 \times \\ & \times \frac{\left( \frac{K - \delta}{4}, W + \frac{\mu}{4}; \frac{K + \delta}{4}, -W + \frac{\mu}{4} \middle| \frac{K}{2} - \kappa; \frac{\mu}{2} \right) (-1)^{\frac{K + n - \delta}{4} - \frac{\nu}{2} + \kappa}}{\left( \frac{j_1}{2} + m, \frac{\mu + \delta}{4}; \frac{j_2}{2} + n - m, \frac{\mu - \delta}{4} \middle| \frac{K}{2}; \frac{\mu}{2} \right) 2^{K/4}} \times \\ & \times \frac{\Delta_{0\mu}^{(J)} \Delta_{\delta/2\nu}^{((K/2) - \kappa)}}{\Delta_{K/2, \mu/2}^{(K/2)}} \left[ \frac{(j_1 + 2m)!(j_2 + 2n - 2m)!}{(K + \kappa + 1)!\kappa!} \right]^{1/2} \binom{n + j_1 + \frac{1}{2}}{m} \binom{n + j_2 + \frac{1}{2}}{n - m} \times \\ & \times D_{\nu, \mu/2}^{(K/2) - \kappa}(\lambda, a, 0) D_{\mu, M}^J(\varphi_1 \ominus \varphi_2), \end{aligned} \quad (164)$$

where  $A_{JM}$  consists of normalization constants and Clebsch–Gordan coefficients.

The solutions of the eigenvalue equations for  $K$  and  $\Omega$  have to be linear combinations of these functions:

$$\Phi_{M\nu}^J = \sum_{j_1 j_2} c_{(j_1 j_2)} \Phi_{JM\nu}^{j_1 j_2}(\vec{\xi}, \vec{\eta}), \quad (165)$$

where  $(j_1 j_2)$  will run over each pair of values which can give such a total angular momentum  $J$  that

$$J \leq j_1 + j_2 \leq K. \quad (166)$$

Looking at the structure of the coefficient, it is easy to understand that our attempt to determine  $a_\nu(\kappa, \mu)$  directly could not be successful.

## 5. A Symmetrical Basis in the Three-Body Problem

In the theory of representation of the  $O(n)$  group one usually considers the canonical Gelfand-Zeitlin chain  $O(n) \supset O(n-1) \supset \dots$ . Such a chain is, however, inconvenient,

when the subject is the many-body problem where the chain  $O(n) \supset O(n-3) \supset \dots$  is more natural. This chain corresponds to the decrease of the number of particles by one ( *i.e.* that of the degree of freedom by three). In addition, in the many-body problem it is reasonable to follow the schemes in the framework of which the angular momenta are summed up, and make sure that the total angular momentum  $J$  and its projection  $M$  remain conserved at all rotations. Of course, the number of group parameters (*i.e.* that of the Euler angles) will decrease.

Indeed, there is no need to consider all the rotations. It is sufficient to take into account only those which do not mix up the components of different vectors.

In the three particle problem the two vectors  $\eta$  and  $\xi$  form the  $O(6)$  group. Here, in order to include the angular momentum  $J$  into the number of observables, one has to separate the  $O(3)$  group. If so, there remain only 5 angles of the 15 of the  $O(6)$  group; they characterize the position of vectors  $\eta$  and  $\xi$  in the six-dimensional space. Three angles define the situation of the plane of the vectors  $\eta$ ,  $\xi$  in the three-dimensional space, two angles determine the angle between these two vectors and their lengths (with the condition  $\xi^2 + \eta^2 = \rho^2$ ). In several works,<sup>31, 32, 34, 43, 45, 50</sup> attempts were made to construct total sets of eigenfunctions for the three-body problem. This was based on the invariance of the Laplacian under  $O(6)$ . The explicit calculations were carried out differently. One of the possibilities is that the eigenfunctions correspond to the classification of the three-particle states according to the chain  $O(3) \supset SU(3) \supset O(6)$  which is characterized by five quantum numbers. Four of them –  $K$ ,  $J$ ,  $M$ ,  $\nu$  – are the six-dimensional angular momentum, corresponding to the eigenvalue of the 6-Laplacian, the usual three-dimensional momentum and its projection, and a number, characterizing the permutation symmetry. Generally speaking, a number of states of the system belongs to the given set  $K$ ,  $J$ ,  $M$ ,  $\nu$ . The fifth quantum number  $\Omega$  which solves this problem is the eigenvalue of the hermitian operator  $\hat{\Omega}$ , commuting with the  $O(3)$  generators. The operator was introduced by Racah;<sup>51</sup> its explicit form was obtained in Descartes coordinates by Badalyan, in polar coordinates it is given in Ref. 32. In order to find the eigenfunctions  $\Phi_{\nu\Omega}^{KJM}$  corresponding to the above choice of quantum numbers, we have to carry out the common solution of the complicated equations  $\Delta\Phi = -K(K+4)\Phi$  and  $\hat{\Omega}\Phi = \Omega\Phi$ .

A different way of constructing the set of functions is based on the graphical tree method. With the help of a simple algorithm a function with quantum numbers  $K, J, M, j_1, j_2$  is built up. If so,  $\Phi_{KJM}^{j_1j_2}$  are, by definition, eigenfunctions of the six-dimensional Laplacian  $O(6)$ ; they do not have, however, definite permutation symmetry properties. Hence, we have to turn from the obtained set of eigenfunctions to functions which change simply when the coordinates of the particles are permuted, *i.e.* to the system with the quantum numbers  $K, J, M, \nu, \Omega$ .

We could not calculate explicitly the transformation coefficients yet. However, making use of the result obtained in Ref. 34, one can carry out a transformation which leads to a system with quantum numbers  $K, J, M, \nu, (j_1j_2)$ . Generally speak-

ing, this system is not orthonormalized yet. It can be orthonormalized either by a standard method, or by the construction of eigenfunctions of the operator  $\hat{\Omega}$  from the function  $\Phi_{KJM}^{\nu(j_1 j_2)}$ . The solution of the corresponding equations is not a difficult task. However, these equations turn out to be high order equations. Thus the solutions of these equations can hardly be presented explicitly.

The obtained solution provides a natural description for the basis of three particles.

### 5.1. Basis functions

Systems of basis functions characterized by different quantum numbers correspond to different parametrizations. What concerns the eigenfunctions symmetrized over permutations, it is convenient to consider them in the  $z, z^*$  space, while the functions  $\Phi_{KJM}^{j_1 j_2}$  are determined in the space of  $\eta$  and  $\xi$ . The latter are well known as the tree functions, see the previous sections. Let us recall notations:

$$\begin{aligned}\Phi_{KJM}^{j_1 j_2}(\eta, \xi) &= \sum_{m_1+m_2=M} C_{j_1 m_1 j_2 m_2}^{JM} \Phi_K^{j_1 j_2 m_1 m_2}(\eta, \xi) = Y_{JM}^{j_1 j_2}(m, n) \Psi_{K j_1 j_2}(\Phi). \\ Y_{JM}^{j_1 j_2}(m, n) &= \sum_{m_1 m_2} C_{j_1 m_1 j_2 m_2}^{JM} Y_{j_1 m_1}(m) Y_{j_2 m_2}(n),\end{aligned}\quad (167)$$

Introducing  $\cos \Phi = \xi$ ,  $\sin \Phi = \eta$  we can write:

$$\begin{aligned}\Psi_{K j_1 j_2}(\Phi) &= N_{K j_1 j_2} (\sin \Phi)^{j_1} (\cos \Phi)^{j_2} P_{(K-j_1-j_2)/2}^{(j_1+1/2, j_2+1/2)}(\cos 2\Phi), \\ N_{K j_1 j_2} &= \left[ \frac{2(K+2)\Gamma((K-j_1-j_2)/2+1)\Gamma((K+j_1+j_2)/2+2)}{\Gamma((K-j_1+j_2)/2+3/2)\Gamma((K+j_1-j_2)/2+3/2)} \right]^{1/2}.\end{aligned}\quad (168)$$

Instead of the Jacobi polynomial we can introduce the Wigner  $d$ -function. Carrying out the transition from the Jacobi polynomial to the Wigner  $d$ -function, it is worth mentioning that there exist different notations. Indeed, in Ref. 46 the definition

$$P_k^{\alpha, \beta}(\cos 2\theta) = i^{b-1} \left[ \frac{(l-b)!(l+b)!}{(l-a)!(l+a)!} \right]^{1/2} (\sin \theta)^{b-a} (\cos \theta)^{-b-a} d_{ab}^l(\cos 2\theta). \quad (169)$$

is given. Here  $i$  appears because the unitary matrices are defined as

$$u(\varphi, \theta, \psi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}, \quad (170)$$

where the second matrix is a unitary one while in Ref. 52 it is orthogonal:

$$u(\varphi, \theta, \psi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}. \quad (171)$$

According to the definition in Ref. 52,

$$\begin{aligned}P_k^{\alpha, \beta}(\cos 2\theta) &= (-1)^{b-a} \left[ \frac{(l-b)!(l+b)!}{(l-a)!(l+a)!} \right]^{1/2} (\sin \theta)^{b-a} (\cos \theta)^{-b-a} d_{ab}^l(\cos 2\theta), \\ l &= k + \frac{\alpha + \beta}{2}, \quad a = \frac{\alpha + \beta}{2}, \quad b = \frac{\beta - \alpha}{2}.\end{aligned}\quad (172)$$

In terms of the  $d$ -function the eigenfunction (167) obtains the form

$$\begin{aligned} \Phi_{KJM}^{j_1 j_2}(\eta, \xi) &= 2(K+2)^{1/2} (-1)^{-j_1-1/2} \\ &\times \sum_{m_1 m_2} C_{j_1 m_1 j_2 m_2}^{JM} Y_{j_1 m_1}(m) Y_{j_2 m_2}(n) \frac{1}{(\sin 2\Phi)^{1/2}} d_{(j_1+j_2+1)/2, (j_2-j_1)/2}^{(K+1)/2}(\cos 2\Phi). \end{aligned} \quad (173)$$

The factor  $(\sin 2\Phi)^{-1/2}$  appears due to the different normalizations over the angle  $\Phi$  in the Jacobi polynomial and in the  $d$ -function. The Jacobi polynomial is normalized in the six-dimensional space, the element of the volume is  $\cos^2 \Phi \sin^2 \Phi d\Phi = \frac{1}{4} \sin^2 2\Phi d\Phi$  while for the  $d$ -function, normalized in the usual space, we have  $\sin 2\Phi d\sin 2\Phi$ .

The transition to the basis function constructed on the vector pair  $\eta', \xi'$  which is related to  $\eta, \xi$  by the transformation

$$\begin{pmatrix} \eta' \\ \xi' \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \quad (174)$$

can be carried out with the help of the coefficient  $\langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\varphi$  obtained in Ref. 53 and Ref. 44:

$$\Phi_{KJM}^{j_1 j_2}(\eta', \xi') = \sum_{j'_1 j'_2} \langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\varphi \Phi_{KJM}^{j'_1 j'_2}(\eta, \xi). \quad (175)$$

We see that for the transition of the basis function from  $\eta, \xi$  to the function of the vector  $z, z^*$  the coefficient  $\langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\varphi$  has to be used at the value  $\varphi = \pi/4$ , after substituting  $\xi$  by  $\zeta$ . In the following it will be shown how the obtained function  $\Phi_{KJM}^{j_1 j_2}(z, z^*)$  can be transformed into  $\Phi_{KJM}^{\nu(j_1 j_2)}(z, z^*)$ .

But let us first investigate the transformation coefficient (175) in detail.

## 5.2. Transformation coefficients $\langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\varphi$

In Ref. 53 the transformation coefficient is defined as the overlap integral of  $\Phi_{KJM}^{j'_1 j'_2}(\eta, \xi)$  and  $\Phi_{KJM}^{j_1 j_2}(\eta', \xi')$ :

$$\begin{aligned} \langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\varphi &= \\ &= \sum_{\substack{m_1 m_2 \\ m'_1 m'_2}} C_{j_1 m_1 j_2 m_2}^{JM} C_{j'_1 m'_1 j'_2 m'_2}^{JM} \int \left( \Phi_K^{j'_1 j'_2 m'_1 m'_2}(\eta, \xi) \right)^* \Phi_K^{j_1 j_2 m_1 m_2}(\eta', \xi') d\eta d\xi. \end{aligned} \quad (176)$$

As it was already mentioned, the transformation takes place at given  $J$  and  $M$  values. The explicit form of the coefficient is calculated with the help of the production function for  $\Phi_K^{j_1 j_2 m_1 m_2}(\eta\xi)$ . The calculation is carried out in two steps.<sup>44</sup> First we consider the coefficient corresponding to the representation in which the vector  $\eta$  is expanded over the “new” vectors  $\xi'$  and  $\eta'$ . After that, we do the same for  $\xi$ . As it is presented in Ref. 44, the transformation coefficient, corresponding to the transition

$\Phi_{K_1 j_1 m_1}^{j_1 0}(\eta', 0)$  to  $\Phi_{K_1 j_1 m_1}^{pq}(\eta', \xi')$ , has the form:

$$\begin{aligned} \langle pq | j_1 0 \rangle_{K j_1 m_1}^\varphi &= (-1)^{(K_1+j_1)/2} 2^{(K_1-j_1)/2} \times \\ &\times \left[ \frac{\left(\frac{K_1-p-q}{2}\right)! \left(\frac{K_1+p+q}{2}\right)! \left(\frac{K_1-j_1}{2}\right)! (K_1+1)!!}{(K_1-p+q+1)!! (K_1+p-q+1)!! \left(\frac{K_1+j_1}{2}+1\right)! (K_1-j_1+1)!!} \right]^{1/2} \\ &\times \begin{pmatrix} p & q & j_1 \\ 0 & 0 & 0 \end{pmatrix} [(2p+1)(2q+1)]^{1/2} (\cos \varphi)^p (\sin \varphi)^q P_{(K_1-p-q)/2}^{(p+1/2, q+1/2)}(-\cos 2\varphi) \quad (177) \end{aligned}$$

and is in fact the tree function in  $O(6)$ . The expression (177) is similar to the formula describing the  $O(2)$ -rotation, which transforms the Legendre-polynomial with the help of the function  $D_{0M}^J = P_{JM}$ , being the tree function in  $O(2)$ . The same procedure has to be carried out for the coefficient of the transformation of  $\Phi_{K_2 j_2 m_2}^{0 j_2}(0, \xi)$  into  $\Phi_{K_2 j_2 m_2}^{0r}(\eta' \xi')$ ; after that, collecting the momenta  $p+r = j'_1$  and  $q+s = j'_2$  (with the total momentum  $J$  and a given  $K = K_1 + K_2$ ), we obtain the general expression for the coefficient

$$\begin{aligned} \langle j'_1 j'_2 | j_1 j_2 \rangle_{K J M}^\varphi &= \frac{\pi}{4} (-1)^{J+(K+j_1+j_2)/2} \times \\ &\times \left( \frac{K_1-j_1}{2} \right)! \left( \frac{K_1+j_1+1}{2} \right)! \left( \frac{K_2-j_2}{2} \right)! \left( \frac{K_2+j_2+1}{2} \right)! \\ &\times \left[ \frac{\left(\frac{K-j'_1-j'_2}{2}\right)! \left(\frac{K+j'_1+j'_2}{2}+1\right)! (K+j'_1+j'_2+1)!! (K+j'_1-j'_2+1)!!}{\left(\frac{K-j_1-j_2}{2}\right)! \left(\frac{K-j_1+j_2}{2}+1\right)! (K-j_1+j_2+1)!! (K+j_1-j_2+1)!!} \right]^{1/2} \\ &\times \sum_{prqs} \left\{ \begin{pmatrix} p & r & j'_1 \\ q & s & j'_2 \\ j_1 & j_2 & J \end{pmatrix} \right\} \left[ \Gamma \left( \frac{K_1-p+q}{2} + \frac{3}{2} \right) \Gamma \left( \frac{K_1+p-1}{2} + \frac{3}{2} \right) \right. \\ &\times \Gamma \left( \frac{K_2-r+s}{2} + \frac{3}{2} \right) \Gamma \left( \frac{K_2+r-s}{2} + \frac{3}{2} \right) \left. \right]^{-1} (\cos \varphi)^{p+s} (\sin \varphi)^{q+r} \\ &\times P_{(K_1-p-q)/2}^{(p+1/2, q+1/2)}(-\cos 2\varphi) P_{(K_2-s-r)/2}^{(s+1/2, r+1/2)}(-\cos 2\varphi). \quad (178) \end{aligned}$$

Here we introduce the notation

$$\begin{aligned} \left\{ \begin{pmatrix} p & r & j'_1 \\ q & s & j'_2 \\ j_1 & j_2 & J \end{pmatrix} \right\} &= [(2j_1+1)(2j_2+1)(2j'_1+1)(2j'_2+1)]^{1/2} (2p+1)(2q+1)(2s+1) \\ &\times (2r+1) \begin{pmatrix} p & q & j_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s & r & j_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p & r & j'_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q & s & j'_2 \\ 0 & 0 & 0 \end{pmatrix} \left\{ \begin{pmatrix} p & r & j'_1 \\ q & s & j'_2 \\ j_1 & j_2 & J \end{pmatrix} \right\}, \quad (179) \end{aligned}$$

which underlines the way how the momenta are transformed.

In Ref. 53 and Ref. 44 not only the technique of the calculations differs but also the form of the final expressions. Thus it is reasonable to find the explicit connection between the two forms of the transformation coefficients  $\langle j'_1 j'_2 | j_1 j_2 \rangle_{K J M}^\phi$  and prove their analogousness. Just this was done in Ref. 34, where a simpler way of obtaining

this coefficient was presented, namely, the substitution of the overlap integral by the matrix element of an exponential function which in fact equals unity.

Instead of calculating the matrix element from the exponent, we expand it in a series over the orders of  $\cos \phi$  and  $\sin \phi$  and transform this series into one over the Jacobi polynomials, *i.e.* the eigenfunctions of the Laplacian. It is convenient to carry out this expansion in two steps: first expand the exponent over the spherical Bessel functions and then collect these functions in Jacobi polynomials. Since this way of calculating the transformation coefficient may turn out to be useful also for other coefficients in the theory of angular momenta, we present it below in detail.

Let us consider the six-dimensional vector

$$\begin{pmatrix} P_i \\ i Q_i \end{pmatrix}. \quad (180)$$

We calculate the matrix element of the exponential function

$$\exp \left[ -2(P_i^2 + Q_i^2) \right] \quad (181)$$

over  $J$  and  $M$ , with the condition  $P_i^2 - Q_i^2 = 0$ . Let us transform one of the multiplication factors in  $P_i^2$  and  $Q_i^2$ , expressing  $P_i$  and  $Q_i$  in terms of  $P_k$  and  $Q_k$  with the help of the transformation (174). Then

$$\begin{aligned} & \langle j'_1 j'_2 | \exp \left[ -2(P_i^2 + Q_i^2) \right] j_1 j_2 \rangle_{KJM} = \\ & = \langle j'_1 j'_2 | \exp \left[ -2P_i P_k \cos \varphi - 2Q_i P_k \cos \varphi + 2iQ_i P_k \sin \varphi + 2iP_i Q_k \sin \varphi \right] | j_1 j_2 \rangle_{KJM}. \end{aligned} \quad (182)$$

The matrix element can be written as

$$\begin{aligned} & \sum_{m_1 m_2} C_{j_1 m j_2 m}^{JM} C_{j'_1 m'_1 j'_2 m'_2}^{JM} \\ & \times \int \exp \left[ -2P_i P_k \cos \varphi - 2Q_i Q_k \cos \varphi + 2iQ_i P_k \sin \varphi + 2iP_i Q_k \sin \varphi \right] \\ & \times Y_{j'_1 m'_1}^*(\hat{P}_k) Y_{j'_2 m'_2}^*(\hat{Q}_i) d\hat{P}_i d\hat{Q}_i d\hat{P}_k d\hat{Q}_k, \end{aligned} \quad (183)$$

where  $\hat{P}_i$ ,  $\hat{Q}_i$ ,  $\hat{P}_k$  and  $\hat{Q}_k$  are the usual three-dimensional polar angles. (Let us mention that the expression (183) coincides up to the normalization with the formula for the transformation coefficient in Ref. 53.) If we now extract in this sum the term with eigenfunctions characterized by the quantum numbers  $j_1, j_2, j'_1, j'_2$ , it turns out to be equal, up to the normalization, to the coefficient we are looking for. Indeed, the overlap integral is, by definition, a matrix element of unity in a mixed representation.

Further, we apply the well-known expansion of the plane wave

$$e^{ipx} = \sum_{\lambda_\mu} i^\lambda j_\lambda(px) Y_{\lambda_\mu}^*(p) Y_{\lambda_\mu}(x) \quad (184)$$

and, expanding into a series the exponent in the integrand (183), we arrive at a rather long, but simple expression

$$\begin{aligned}
 & \sum_{\substack{m_1 m_2 \\ m'_1 m'_2}} C_{j_1 m_1 j_2 m_2}^{JM} C_{j'_1 m'_1 j'_2 m'_2}^{JM} \int \sum_{\substack{p r q s \\ \pi \rho \kappa \sigma}} (-1)^{(p+r+q+s)/2} j_p (2i P_i P_k \cos \varphi) j_r (2Q_i P_k \sin \varphi) \\
 & \times j_q (2P_i Q_k \sin \varphi) j_s (2i Q_i Q_k \cos \varphi) Y_{p\pi}^*(\hat{P}_i) Y_{q\kappa}^*(\hat{P}_i) Y_{j_1 m_1}(\hat{P}_1) Y_{r\rho}^*(Q_i) Y_{s\sigma}^*(Q_i) \times \\
 & \times Y_{j_2 m_2}(Q_i) Y_{p\pi}(\hat{P}_k) Y_{r\rho}(\hat{P}_k) Y_{j'_1 m'_1}^*(\hat{P}_k) Y_{q\kappa}(\hat{Q}_k) Y_{s\sigma}(\hat{Q}_k) Y_{j'_2 m'_2}^*(\hat{Q}_k) d\hat{P}_i d\hat{Q}_i d\hat{P}_k d\hat{Q}_k.
 \end{aligned} \tag{185}$$

Making use of the well-known features of the spherical functions

$$Y_{lm}^*(\vartheta, \varphi) = (-1)^m Y_{l-m}(\vartheta, \varphi) \tag{186}$$

and

$$\int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \sin \vartheta Y_{l_1 m_1}(\vartheta, \varphi) Y_{l_2 m_2}(\vartheta, \varphi) = \left[ \frac{(2l_1 + 1)(2l_2 + 1)}{4\pi(2l_3 + 1)} \right]^{1/2} C_{l_1 0 l_2 0}^{l_3 0} C_{l_1 m_1 l_2 m_2}^{l_3 m_3} \tag{187}$$

(see Ref. 52), we re-write (185) in the form

$$\begin{aligned}
 & \frac{\pi^2}{16} \sum_{\substack{p r q s \\ \pi \rho \kappa \sigma}} \sum_{\substack{m_1 m_2 \\ m'_1 m'_2}} C_{j_1 m_1 j_2 m_2}^{JM} C_{j'_1 m'_1 j'_2 m'_2}^{JM} C_{p\pi q\kappa}^{j_1 m_1 j_2 m_2} C_{r\rho s\sigma}^{j'_1 m'_1 j'_2 m'_2} C_{p\pi r\rho}^{j_1 m_1 j_2 m_2} C_{q\kappa s\sigma}^{j'_1 m'_1 j'_2 m'_2} C_{p0q0}^{j_1 0 j_2 0} C_{r0s0}^{j'_1 0 j'_2 0} C_{p0r0}^{j_1 0 j_2 0} C_{q0s0}^{j'_1 0 j'_2 0} \\
 & \times \frac{(2p+1)(2r+1)(2q+1)(2s+1)}{[(2j_1+1)(2j_2+1)(2j'_1+1)(2j'_2+1)]^{1/2}} j_p (2i P_i P_k \cos \varphi) \times \\
 & \times j_r (2Q_i P_k \sin \varphi) j_p (2P_i Q_k \sin \varphi) j_s (2i Q_i Q_k \cos \varphi) (-1)^{m_1+m_2-j_1-j_2+1/2(p+r+q+s)}.
 \end{aligned} \tag{188}$$

The summation of the Clebsch-Gordan coefficients leads to the  $9j$ -coefficient

$$\begin{aligned}
 & \sum_{m_s m_{sk}} C_{j_1 m_1 j_2 m_2}^{j_{12} m_{12}} C_{j_3 m_3 j_4 m_4}^{j_{34} m_{34}} C_{j_{12} m_{12} j_{34} m_{34}}^{j m} C_{j_1 m_1 j_3 m_3}^{j_{13} m_{13}} C_{j_2 m_2 j_4 m_4}^{j_{24} m_{24}} C_{j_{13} m_{13} j_{24} m_{24}}^{j' m'} = \\
 & = \delta_{jj'} \delta_{mm'} \left[ (2j_{12}+1)(2j_{13}+1)(2j_{24}+1)(2j_{34}+1) \right]^{1/2} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{matrix} \right\}.
 \end{aligned} \tag{189}$$

Hence, in (188) we can get rid of the series of sums, including the  $9j$ -coefficient, separating the quantum numbers  $j_1, j_2, j'_1, j'_2, J$ :



$$\begin{aligned} & \frac{\pi^2}{16} \sum_{prqs} \left\{ \begin{matrix} p & r & j'_1 \\ q & s & j_2 \\ j_1 & j_2 & J \end{matrix} \right\} C_{p0r0}^{j'_1 0} C_{q0s0}^{j'_2 0} C_{p0q0}^{j_1 0} C_{p0s0}^{j_2 0} (2p+1)(2r+1)(2q+1)(2s+1) \\ & \times (-1)^{1/2(p+r+q+s)} j_p(2iP_i P_k \cos \varphi) j_r(2Q_i P_k \sin \varphi) j_q(2P_i Q_k \sin \varphi) j_s(2iQ_i Q_k \cos \varphi). \end{aligned} \quad (190)$$

In the following, we apply the expression for the expansion of the product of two Bessel functions into the sum of hypergeometric functions (see Ref. 54). Here we substitute in the standard formulae the hypergeometric functions by Jacobi polynomials (or Wigner's  $d$ -functions). Substituting also the lengths of the vectors  $P_i$ ,  $P_k$ ,  $Q_i$ ,  $Q_k$  by unity, we can write

$$\begin{aligned} j_p(2i \cos \varphi) j_q(2 \sin \varphi) &= \frac{\pi}{4} \sum_{K_1} P_{(K_1-p-q)/2}^{(p+1/2, q+1/2)}(-\cos 2\varphi) \times \\ &\times \frac{(i)^{K_1-q} (\cos \varphi)^p (\sin \varphi)^q}{\Gamma\left(\frac{K_1+p-q}{2} + \frac{3}{2}\right) \Gamma\left(\frac{K_1-p+q}{2} + \frac{3}{2}\right)} j_s(2i \cos \varphi) j_r(2 \sin \varphi) = \\ &= \frac{\pi}{4} \sum_{K_2} P_{(K_2-s-r)/2}^{(s+1/2, r+1/2)}(-\cos 2\varphi) \frac{(i)^{K_2-r} (\cos \varphi)^s (\sin \varphi)^r}{\Gamma\left(\frac{K_2+s-r}{2} + \frac{3}{2}\right) \Gamma\left(\frac{K_2-p+q}{2} + \frac{3}{2}\right)}. \end{aligned} \quad (191)$$

Selecting from these sums only terms with definite  $K = K_1 + K_2$ , we obtain the expression

$$\begin{aligned} & \frac{1}{(16)^2} \sum_{prqs} \left\{ \left\{ \begin{matrix} p & r & j'_1 \\ q & s & j_2 \\ j_1 & j_2 & J \end{matrix} \right\} \right\} \frac{(-1)^{K/2}}{\Gamma\left(\frac{K_1+p+q}{2} + \frac{3}{2}\right) \Gamma\left(\frac{K_1+p-q}{2} + \frac{3}{2}\right)} \times \\ & \times \left[ \Gamma\left(\frac{K_2-r+s}{2} + \frac{3}{2}\right) \Gamma\left(\frac{K_2+r-s}{2} + \frac{3}{2}\right) \right] \times \\ & \times (\cos \varphi)^{p+s} (\sin \varphi)^{q+r} P_{(K_1-p-q)/2}^{(p+1/2, q+1/2)}(-\cos 2\varphi) P_{(K_2-s-r)/2}^{(s+1/2, r+1/2)}(-\cos 2\varphi). \end{aligned} \quad (192)$$

The formula (192) coincides with the general form of the coefficient  $\langle j'_1 j'_2 | j_1 j_2 \rangle_{KM}^\phi$ , given in Ref. 44.

Let us, finally, present the orthonormalized transformation coefficient in terms of the  $d$ -function which makes the interpretation of different expressions easier with

the help of the six-dimensional rotations:

$$\begin{aligned}
 \langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\varphi &= \frac{\pi}{2} (-1)^{J+1} \left( \frac{K_1 - j_1}{2} \right)! \left( \frac{K_1 + j_1 + 1}{2} \right)! \left( \frac{K_2 - j_2}{2} \right)! \left( \frac{K_2 + j_2 + 1}{2} \right)! \\
 &\times \frac{a_{Kj'_1 j'_2}}{a_{Kj_1 j_2}} \sum_{prqs} \left[ \left( \frac{K_1 - p + q + 1}{2} \right)! \left( \frac{K_1 + p - q + 1}{2} \right)! \left( \frac{K_1 - p - q}{2} \right)! \left( \frac{K_1 + p + q + 2}{2} \right)! \right]^{-\frac{1}{2}} \\
 &\times \left[ \left( \frac{K_2 - s + r + 1}{2} \right)! \left( \frac{K_2 + s - r + 1}{2} \right)! \left( \frac{K_2 - r - s}{2} \right)! \left( \frac{K_2 + r + s}{2} \right)! \right]^{1/2} \times \\
 &\times \left\{ \begin{Bmatrix} p & r & j'_1 \\ q & s & j'_2 \\ j_1 & j_2 & J \end{Bmatrix} \right\} \frac{1}{\sin 2\varphi} d_{q+p+1/2, (p-q)/2}^{(K_1+1)/2}(2\varphi) d_{(r+s+1)/2, (s-r)/2}^{(K_2+1)/2}(2\varphi), \quad (193)
 \end{aligned}$$

where

$$\begin{aligned}
 a_{Kj_1 j_2} &= \\
 &= \left[ \left( \frac{K - j_1 - j_2}{2} \right)! \left( \frac{K + j_1 + j_2}{2} + 1 \right)! \left( \frac{K - j_1 + j_2 + 1}{2} \right)! \left( \frac{K + j_1 - j_2 + 1}{2} + 1 \right)! \right]^{\frac{1}{2}}. \quad (194)
 \end{aligned}$$

Since the normalizations of the Jacobi polynomial and the  $d$ -function are known, we do not have to think about it while carrying out the calculations. (Let us remind once more that the factor  $(\sin 2\phi)^{-1}$  is related to the six-dimensional normalization.)

The expression (193) is similar to the formula of the “three  $d$ -functions” in  $SU(2)$ ; it shows the expansion of one of the  $O(6)$   $d$ -functions over the products of two  $d$ -functions of a special form. As usual in such expressions, there is a freedom in the choice of  $K_1$  or  $K_2$  ( $K_1 + K_2 = K$ ).

### 5.3. Applying $\langle j'_1 j'_2 | j_1 j_2 \rangle_{KLM}^\Phi$ to the three-body problem

Let us use the obtained formulae for the calculation of the coefficients of the transition from  $\Phi_{KJM}^{j'_1 j'_2}(\eta, \xi)$  to  $\Phi_{KJM}^{j_1 j_2}(z, z^*)$ . As it was shown already, this can be done by the application of the transformation coefficient at the  $2\phi = \pi/2$  value, first substituting  $\eta$  by  $i\eta = \zeta$ . The argument of the  $d$ -functions in (193) is  $2\phi$ . The turns by  $\pi/2$  – the so-called Weyl-coefficients – are included in many formulae of the theory of  $O(n)$  representation.) The function of the new arguments

$$\Phi_{KJM}^{j_1 j_2}(z, z^*) = \sum_{j'_1 j'_2} \langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^{\pi/4} \Phi_{KJM}^{j'_1 j'_2}(\zeta, \xi) \quad (195)$$

can be written in the form

$$\begin{aligned}
 \Phi_{KJM}^{j_1 j_2}(z, z^*) &= \frac{\pi}{2} (-1)^{J+1} \left( \frac{K_1 - j_1}{2} \right)! \left( \frac{K_1 + j_1 + 1}{2} \right)! \left( \frac{K_2 - j_2}{2} \right)! \left( \frac{K_2 + j_2 + 1}{2} \right)! \\
 &\times \sum_{\substack{j'_1 j'_2 \\ m'_1 m'_2}} C_{j'_1 m'_1 j'_2 m'_2}^{JM} N_{K j'_1 j'_2} \frac{a_{K j'_1 j'_2}}{a_{K j_1 j_2}} \sum_{prqs} \left[ \left( \frac{K_1 - p + q + 1}{2} \right)! \left( \frac{K_1 + p - q + 1}{2} \right)! \times \right. \\
 &\times \left( \frac{K_1 - p - q}{2} \right)! \left( \frac{K_1 + p + q}{2} + 1 \right)! \left. \right]^{-\frac{1}{2}} \left[ \left( \frac{K_2 - s + r + 1}{2} \right)! \left( \frac{K_2 + s - r + 1}{2} \right)! \times \right. \\
 &\times \left( \frac{K_2 - r - s}{2} \right)! \left( \frac{K_2 + r + s}{2} + 1 \right)! \left. \right]^{-\frac{1}{2}} \left\{ \begin{matrix} p & r & j'_1 \\ q & s & j_2 \\ j_1 & j_2 & J \end{matrix} \right\} d_{(q+p+1)/2, (p-q)/2}^{(K_1+1)/2} \left( \frac{\pi}{2} \right) \times \\
 &\times d_{(r+s+1)/2, (s-r)/2}^{(K_2+1)/2} \left( \frac{\pi}{2} \right) Y_{j'_1 m'_1}(z) Y_{j'_2 m'_2}(z^*) \frac{1}{z^2 - z^{*2}} d_{(j'_1+j'_2+1)/2, (j'_1-j'_2)/2}^{(K+1)/2} (z^2 + z^{*2}),
 \end{aligned} \tag{196}$$

where the relations

$$\begin{aligned}
 \frac{1}{2} (z^2 + z^{*2}) &= \xi^2 - \eta^2, \quad z z^* = \xi^2 + \eta^2. \\
 z^2 - z^{*2} &= 2i \sin 2\Phi = 4i\xi\eta,
 \end{aligned} \tag{197}$$

are taken into account.

As it was mentioned in the previous section, it is convenient to use  $K_2 = r + s$  in (196).

The expression (196) presents a series over the degrees of  $z$  and  $z^*$ . In each term of this series the degree of  $z$  is  $p + q$ , that of  $z^* - r + s$ . Considering the parametrization  $z$ , we see that each term of  $z$  and  $z^*$  introduces a factor  $\exp(-i\lambda/2)$  and  $\exp(i\lambda/2)$ , respectively. Because of this, any term in the series (196) will contain a factor  $\exp(-i\lambda(p + q - r - s)/2)$ . The series can be changed into a Fourier series over  $\exp(-i\nu\lambda)$ , if collecting all terms of the series with a given

$$p + q - r - s = 2\nu. \tag{198}$$

In this case each term of the new series will have a definite value of  $\nu$  and, hence, the Fourier series will be at the same time a series over the eigenfunctions of the operator  $N$ . We arrive at functions characterized by the set  $K, J, M, \nu, (j_1 j_2)$ . Their normalization can be easily obtained from that of the  $d$ -functions. The only remaining problem is the transition from  $(j_1 j_2)$  to  $\Omega$ , which we have already mentioned above. Although a multiplicity of the equations appears practically only at large  $K$ , the construction of convenient expressions deserves further efforts.

#### 5.4. The $d$ -function of the $O(6)$ group

The coefficient  $\langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\phi$  describes the rotations in the six-dimensional space. From these rotations one can, obviously, construct an arbitrary rotation. The  $O(n)$

rotations are usually composed of  $n(n-1)/2$  rotations on all coordinate planes.<sup>55</sup> With the help of the calculated coefficients rotations can be constructed in any plane, characterized by two arbitrary vectors  $\eta$  and  $\xi$ . In other words, these coefficients lead to simultaneous rotations in three two-dimensional planes  $(\eta_x, \xi_x)$ ,  $(\eta_y, \xi_y)$  and  $(\eta_z, \xi_z)$  and simplify the calculations seriously.

It would be rather interesting to generalize all this to the group  $O(n)$  ( $n > 6$ ), and consider tensors of higher dimensions instead of vectors.

## 6. Symmetries in the Classical Three-Body Problem

Quantum mechanical three-body systems possess the symmetry of motion of a five-dimensional sphere with respect to both the free motion and elastic forces (see for example Refs. 32, 37, 40). As the quantum mechanical problem obviously has the same symmetry as the classical one, it seems to be worthwhile to consider the classical equations of motion from this point of view.

Arbitrary motions of a three-body system can be described as rotations and deformations of a triangle formed by the three particles: the equations of motion of the triangle turn out to be very similar to the equations of two coupled tops, one of them reflecting the hidden (non-geometrical) symmetry of the deformative motion of the triangle.

We collect here different types of equations and formulae connected with the classical three-body problem.

### 6.1. Examples of non-rotating triangles

In dealing with a three-particle system, let us first recall the used here system of coordinates. The radius vectors  $\vec{x}_i$  ( $i = 1, 2, 3$ ) of the three particles are fixed by the condition

$$\vec{x}_1 + \vec{x}_2 + \vec{x}_3 = 0. \quad (199)$$

The Jacobi coordinates  $\vec{\xi}$  and  $\vec{\eta}$  are given in the case of equal masses in the form

$$\begin{aligned} \vec{\xi} &= -\sqrt{\frac{3}{2}}(\vec{x}_1 + \vec{x}_2), \\ \vec{\eta} &= \frac{1}{\sqrt{2}}(\vec{x}_1 - \vec{x}_2), \\ \xi^2 + \eta^2 &= x_1^2 + x_2^2 + x_3^2 = \varrho^2, \end{aligned} \quad (200)$$

where  $\varrho$  is the radius of the five-dimensional sphere. Further, we introduce the complex vector

$$\begin{aligned} \vec{z} &= \vec{\xi} + i\vec{\eta}, \\ \vec{z}^* &= \vec{\xi} - i\vec{\eta}. \end{aligned} \quad (201)$$

Consider now a triangle with vertices  $x_1, x_2, x_3$ . The position of this triangle in space is characterized by the vectors  $\vec{l}_1$  and  $\vec{l}_2$ , which together with the vector  $\vec{l} = \vec{l}_1 \times \vec{l}_2$

form the moving coordinate system. They are connected with vectors  $z$  and  $z^*$  in the following way:

$$\begin{aligned}\vec{z} &= \frac{\varrho}{\sqrt{2}} e^{-i(\lambda/2)} \left( e^{i(a/2)} \vec{l}_1 + i e^{-i(a/2)} \vec{l}_2 \right), \\ \vec{z}^* &= \frac{\varrho}{\sqrt{2}} e^{i(\lambda/2)} \left( e^{-i(a/2)} \vec{l}_1 - i e^{i(a/2)} \vec{l}_2 \right),\end{aligned}\quad (202)$$

where  $0 \leq a \leq \pi$ ,  $0 \leq \lambda \leq 2\pi$ . The variables  $\lambda$  and  $a$  determine the form of the triangle. Using expressions (202), we can write  $\vec{\xi}$  and  $\vec{\eta}$  in the form

$$\begin{aligned}\vec{\xi} &= \frac{\varrho}{\sqrt{2}} \left( \cos \frac{a-\lambda}{2} \vec{l}_1 + \sin \frac{a+\lambda}{2} \vec{l}_2 \right), \\ \vec{\eta} &= \frac{\varrho}{\sqrt{2}} \left( \sin \frac{a-\lambda}{2} \vec{l}_1 + \cos \frac{a+\lambda}{2} \vec{l}_2 \right).\end{aligned}\quad (203)$$

This means that we can consider  $\vec{\xi}$  and  $\vec{\eta}$  as a result of two transformations

$$\begin{bmatrix} \vec{\xi} \\ \vec{\eta} \end{bmatrix} = \frac{\varrho}{\sqrt{2}} \begin{bmatrix} \cos \frac{\lambda}{2} & \sin \frac{\lambda}{2} \\ -\sin \frac{\lambda}{2} & \cos \frac{\lambda}{2} \end{bmatrix} \begin{bmatrix} \cos \frac{a}{2} & \sin \frac{a}{2} \\ \sin \frac{a}{2} & \cos \frac{a}{2} \end{bmatrix} \begin{bmatrix} \vec{l}_1 \\ \vec{l}_2 \end{bmatrix}.$$
 (204)

To make the picture clearer, let us consider the case of a non-rotating triangle. We need for this purpose the expressions

$$\begin{aligned}\xi^2 &= \frac{\varrho^2}{2} (1 + \sin a \sin \lambda), \\ \eta^2 &= \frac{\varrho^2}{2} (1 - \sin a \sin \lambda), \\ \xi \vec{\eta} &= \frac{\varrho^2}{2} \sin a \cos \lambda.\end{aligned}\quad (205)$$

The angle  $\theta$  between vectors  $\vec{\xi}$  and  $\vec{\eta}$

$$\vec{\xi} \vec{\eta} = |\xi| |\eta| \cos \theta \quad (206)$$

can be written in terms of our variables as

$$\cos \theta = \frac{\cos \lambda \sin a}{\sqrt{1 - \sin^2 a \sin^2 \lambda}}. \quad (207)$$

Note that the components of the moment of inertia are

$$\varrho^2 \sin^2 \left( \frac{a}{2} - \frac{\pi}{2} \right), \quad \varrho^2 \cos^2 \left( \frac{a}{2} - \frac{\pi}{4} \right), \quad \varrho^2. \quad (208)$$

It is obvious that if  $a = \text{const}$ , the variations of  $\lambda$  lead to deformations of the triangle which do not affect the values of the momenta of inertia. We can write

$$\begin{aligned}|\xi| &= \frac{\varrho}{\sqrt{2}} \sqrt{1 + C \sin \lambda}, \\ |\eta| &= \frac{\varrho}{\sqrt{2}} \sqrt{1 - C \sin \lambda}, \\ \cos \theta &= \frac{C \cos \lambda}{\sqrt{1 - C^2 \sin^2 \lambda}},\end{aligned}\quad (209)$$

where  $C = \sin a$ .

For example, if  $a = 0$  (*i.e.*  $C = 0$ ), we have

$$|\xi| = \frac{\varrho}{\sqrt{2}}, \quad |\eta| = \frac{\varrho}{\sqrt{2}}, \quad \cos \theta = 0. \quad (210)$$

In this case vectors  $\vec{\xi}$  and  $\vec{\eta}$  are orthogonal independently of the value of  $\lambda$ , and only similarity transformations of the triangle are possible. On the other hand, if  $a = \pi/2$ , then

$$\begin{aligned} |\xi| &= \frac{\varrho}{\sqrt{2}} \sqrt{1 + \sin \lambda}, \\ |\eta| &= \frac{\varrho}{\sqrt{2}} \sqrt{1 - \sin \lambda}, \\ \cos \theta &= 1, \end{aligned} \quad (211)$$

*i.e.* the system is linear and the ends of the vectors  $\vec{\xi}$  and  $\vec{\eta}$  are oscillating about the point  $\varrho/\sqrt{2}$ . Expressing the positions of all three particles in the c.m. system in terms of  $\vec{\xi}$  and  $\vec{\eta}$ :

$$\begin{aligned} \vec{x}_1 &= \frac{1}{\sqrt{6}}\vec{\xi} + \frac{1}{\sqrt{2}}\vec{\eta} = \sqrt{\frac{2}{3}} \left[ \cos \frac{2\pi}{3} \vec{\xi} + \sin \frac{2\pi}{3} \vec{\eta} \right], \\ \vec{x}_2 &= -\frac{1}{\sqrt{6}}\vec{\xi} - \frac{1}{\sqrt{2}}\vec{\eta} = \sqrt{\frac{2}{3}} \left[ \cos \frac{4\pi}{3} \vec{\xi} + \sin \frac{4\pi}{3} \vec{\eta} \right], \\ \vec{x}_3 &= \sqrt{\frac{2}{3}} \vec{\xi} \end{aligned} \quad (212)$$

it will be easy to represent the position of the particles by their radius vectors. As an illustration, we consider the case  $a = \pi/2$  for different values of  $\lambda$ :

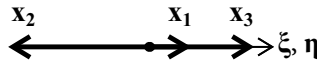


Fig. 2.  $\alpha = \pi/2$ ,  $\lambda = 0$

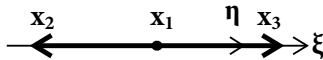


Fig. 3.  $\alpha = \pi/2$ ,  $\lambda = \pi/6$

Consider now those deformations which are connected with the change of  $a$  *i.e.* those which do not leave the moment of inertia unaltered.

Let  $\lambda = 0$ , then

$$|\xi| = \frac{\varrho}{\sqrt{2}}, \quad |\eta| = \frac{\varrho}{\sqrt{2}}, \quad \sin a = \cos \theta. \quad (213)$$

The angle between  $\vec{\xi}$  and  $\vec{\eta}$  is

$$\theta = \frac{\pi}{2} - a.$$

We list here a few special cases:

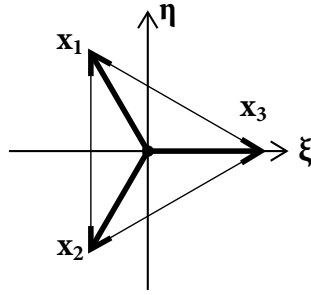
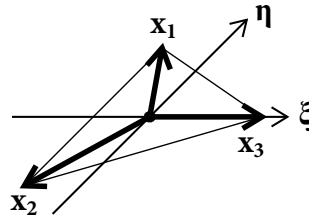


Fig. 4.  $\alpha = 0, \quad \lambda = 0$



$\alpha = \pi/4, \quad \lambda = 0$

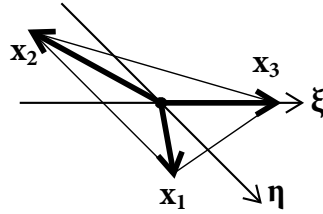
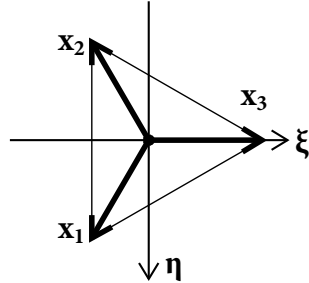


Fig. 5.  $\alpha = 3\pi/4, \quad \lambda = 0$



$\alpha = \pi, \quad \lambda = 0$

Considering the case  $\lambda = \pi/2$ , from the formulae

$$\begin{aligned} |\xi| &= \frac{\varrho}{\sqrt{2}} \sqrt{1 + \sin a}, \\ |\eta| &= \frac{\varrho}{\sqrt{2}} \sqrt{1 - \sin a}, \\ \cos \theta &= 0, \end{aligned} \quad (214)$$

it can be easily seen that  $\vec{\xi}$  and  $\vec{\eta}$  are orthogonal and their lengths can oscillate between zero and  $\varrho$ .

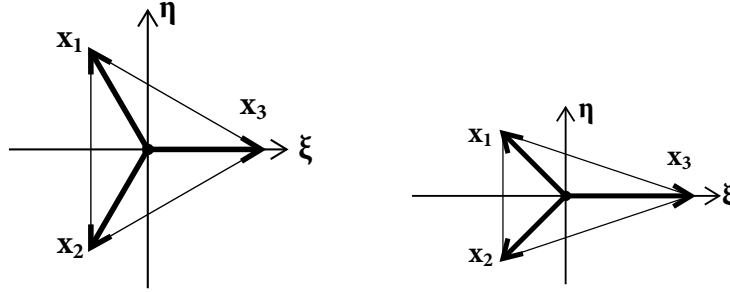


Fig. 6.  $\alpha = 0, \quad \lambda = \pi/2$

$\alpha = \pi/6, \quad \lambda = \pi/2$

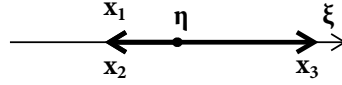


Fig. 7.  $\alpha = \pi/2, \quad \lambda = \pi/2$

## 6.2. The free Lagrangian

In this subsection we present the Euler equations. First of all, we have to construct the Lagrangian  $L = T - U$ . For free particles we have

$$L = T = \frac{1}{2} \left[ \frac{ds}{dt} \right]^2. \quad (215)$$

Let us begin with the expression

$$d\vec{z} = \frac{1}{\varrho} \vec{z} d\varrho - \frac{i}{2} \vec{z} d\lambda + \frac{1}{2} e^{\gamma\lambda} (\vec{l} \times \vec{z})^* da - (d\vec{\omega} \times \vec{z}), \quad (216)$$

where  $d\vec{\omega}$  is the infinitesimal rotation with projections  $d\omega_l$  onto the fixed axes. The rotations about the moving axes are defined as

$$d\Omega_l = \vec{l}_l d\vec{\omega}. \quad (217)$$

They can be expressed in terms of the Euler angles in the form

$$\begin{aligned} d\Omega_1 &= -\cos \varphi_1 \sin \theta d\varphi_2 + \sin \varphi_1 d\theta, \\ d\Omega_2 &= \sin \varphi_1 \sin \theta d\varphi_2 + \cos \varphi_1 d\theta, \\ d\Omega_3 &= -d\varphi_1 - \cos \theta d\varphi_2. \end{aligned} \quad (218)$$

From (19) we get

$$\begin{aligned} ds^2 = |dz|^2 &= \varrho^2 \left[ \frac{1}{4} da^2 + \frac{1}{4} d\lambda^2 + \frac{1}{2} d\Omega_1^2 + \frac{1}{2} d\Omega_2^2 + \right. \\ &\quad \left. + d\Omega_3^2 - \sin \alpha d\Omega_1 d\Omega_2 - \cos \alpha d\Omega_3 d\lambda \right] + d\varrho^2. \end{aligned} \quad (219)$$



Obviously, the wanted expression will be

$$T = \frac{1}{2} \varrho^2 \left[ \frac{1}{4} \dot{a}^2 + \frac{1}{4} \dot{\lambda}^2 + \frac{1}{2} \dot{\Omega}_1^2 + \frac{1}{2} \dot{\Omega}_2^2 + \dot{\Omega}_3^2 - \sin a \dot{\Omega}_1 \dot{\Omega}_2 - \cos a \dot{\Omega}_3 \dot{\lambda} \right] + \frac{1}{2} \dot{\varrho}^2. \quad (220)$$

Due to the formula

$$p_t = \frac{\partial T}{\partial \dot{q}_t}, \quad (221)$$

we can write the momenta

$$\begin{aligned} p_a &= \frac{1}{4} \varrho^2 \dot{a}, \\ p_\lambda &= \frac{1}{2} \varrho^2 \left( \frac{1}{2} \dot{\lambda} - \cos a \dot{\Omega}_3 \right), \\ p_{\Omega_1} &= \frac{1}{2} \varrho^2 (\dot{\Omega}_1 - \sin a \dot{\Omega}_2), \\ p_{\Omega_2} &= \frac{1}{2} \varrho^2 (\dot{\Omega}_2 - \sin a \dot{\Omega}_1), \\ p_{\Omega_3} &= \frac{1}{4} \varrho^2 (2 \dot{\Omega}_3 - \cos a \dot{\lambda}), \\ p_\varrho &= \dot{\varrho}, \end{aligned} \quad (222)$$

and the corresponding  $\dot{p}_i$ :

$$\begin{aligned} \dot{p}_a &= \frac{1}{4} \varrho \dot{\varrho} \dot{a} + \frac{1}{2} \varrho^2 \ddot{a}, \\ \dot{p}_\lambda &= \frac{1}{2} \varrho^2 \left( \frac{1}{4} \ddot{\lambda} + \sin a \dot{a} \dot{\Omega}_3 - \cos a \ddot{\Omega}_3 \right) + \varrho \left( \frac{1}{2} \dot{\lambda} \dot{\varrho} - \cos a \dot{\varrho} \dot{\Omega}_3 \right), \\ \dot{p}_{\Omega_1} &= \frac{1}{2} \varrho^2 (\ddot{\Omega}_1 - \cos a \dot{a} \dot{\Omega}_2 - \sin a \ddot{\Omega}_2) + \varrho (\dot{\varrho} \dot{\Omega}_1 - \sin a \dot{\varrho} \dot{\Omega}_2), \\ \dot{p}_{\Omega_2} &= \frac{1}{2} \varrho (\ddot{\Omega}_2 - \cos a \dot{a} \dot{\Omega}_1 - \sin a \ddot{\Omega}_1) + \varrho (\dot{\varrho} \dot{\Omega}_2 - \sin a \dot{\varrho} \dot{\Omega}_1), \\ \dot{p}_{\Omega_3} &= \frac{1}{2} \varrho^2 (2 \ddot{\Omega}_3 + \sin a \dot{a} \dot{\lambda} - \cos a \ddot{\lambda}) + \varrho (2 \dot{\varrho} \dot{\Omega}_3 - \cos a \dot{\varrho} \dot{\lambda}), \\ \dot{p}_\varrho &= \dot{\varrho}. \end{aligned} \quad (223)$$

We can now construct the equations of motion

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = 0. \quad (224)$$

To obtain the equations explicitly, we have to return to the Euler angles. Indeed, as  $\dot{\Omega}_i$  are not derivatives of any angles  $\Omega_i$  (that is why they are called quasi-coordinates), the Euler equation in terms of these quasi-coordinates must be written in another form.

Thus, instead of (220) we have to take the Lagrangian expressed in terms of the Euler angles:

$$\begin{aligned}
 T = \frac{1}{2} \varrho^2 & \left[ \frac{1}{4} \dot{a}^2 + \frac{1}{4} \dot{\lambda}^2 + \dot{\varphi}_1^2 + \frac{1}{2} \dot{\theta}^2 + \frac{1}{2} \dot{\varphi}_2^2 + \frac{1}{2} \cos^2 \theta \dot{\varphi}_2^2 + 2 \cos \theta \dot{\varphi}_1 \dot{\varphi}_2 \right. \\
 & + \sin a \left( \frac{1}{2} \sin 2\varphi_1 \sin^2 \theta \dot{\varphi}_2^2 + \cos 2\varphi_1 \sin \theta \dot{\varphi}_2 \dot{\theta} - \frac{1}{2} \sin 2\varphi_1 \dot{\theta}^2 \right) \\
 & \left. + \cos a (\dot{\varphi}_1 \dot{\lambda} + \cos \theta \dot{\varphi}_2 \dot{\lambda}) \right] + \frac{1}{2} \dot{\varrho}^2. \quad (225)
 \end{aligned}$$

The equations of free motion are as follows:

$$\begin{aligned}
 \frac{1}{2} \ddot{a} - \cos a & \left( \frac{1}{2} \sin 2\varphi_1 \sin^2 \theta \dot{\varphi}_2^2 + \cos 2\varphi_1 \sin \theta \dot{\varphi}_2 \dot{\theta} - \frac{1}{2} \sin 2\varphi_1 \dot{\theta}^2 \right) \\
 & + \sin a \left( \dot{\varphi}_1 \dot{\lambda} + \cos \theta \dot{\varphi}_2 \dot{\lambda} \right) + \frac{1}{2} \dot{\varphi} \ddot{a} = 0, \quad (226)
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2} \ddot{\lambda} - \sin a & (\dot{a} \dot{\varphi}_1 + \cos \theta \dot{a} \dot{\varphi}_2) + \cos a (\dot{\varphi}_1 \sin \theta \dot{\theta} \dot{\varphi}_2 + \cos \theta \ddot{\varphi}_2) \\
 & + \frac{1}{\varrho} \dot{\varrho} \dot{\lambda} + \frac{2}{\varrho} \cos a (\dot{\varphi}_1 \dot{\varrho} + \cos \theta \dot{\varphi}_2 \dot{\varrho}) = 0, \quad (227)
 \end{aligned}$$

$$\begin{aligned}
 \dot{\varphi}_1 - \sin \theta \dot{\theta} \dot{\varphi}_2 + \cos \theta \dot{\varphi}_2 + \frac{1}{2} \cos a \dot{\lambda} - \\
 - \frac{1}{2} \sin a \left[ \dot{a} \dot{\lambda} + \cos 2\varphi_1 \sin^2 \theta \dot{\varphi}_2^2 - 2 \sin 2\varphi_1 \sin \theta \dot{\varphi}_2 \dot{\theta} - \cos 2\varphi_1 \dot{\theta}^2 \right] \\
 + \frac{1}{\varrho} \left( 2 \dot{\varphi}_1 \dot{\varrho} + 2 \cos \theta \dot{\varphi}_2 \dot{\varrho} + \cos a \dot{\lambda} \dot{\varrho} \right) = 0, \quad (228)
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2} \sin a \cos 2\varphi_1 \ddot{\theta} + \frac{1}{2} \sin \theta (1 + \sin a \sin 2\varphi_1) \ddot{\varphi}_2 - (1 + \sin a \sin 2\varphi_1) \dot{\varphi}_1 \dot{\theta} \\
 + \sin a \left( \cos 2\varphi_1 \sin \theta \dot{\varphi}_1 \dot{\varphi}_2 - \frac{1}{2} \text{ctg} \theta \cos 2\varphi_1 \dot{\theta}^2 + \sin 2\varphi_1 \cos \theta \dot{\varphi}_2 \dot{\theta} \right) \\
 + \cos a \left( \frac{1}{2} \sin 2\varphi_1 \theta \dot{a} \dot{\varphi}_2 + \frac{1}{2} \cos 2\varphi_1 \dot{\theta} \dot{a} - \frac{1}{2} \dot{\lambda} \dot{\theta} \right) \\
 + \frac{1}{\varrho} \left[ \sin \theta \dot{\varphi}_2 \dot{\varrho} (1 + \sin a \sin 2\varphi_1) + \sin a \cos 2\varphi_1 \dot{\theta} \dot{\varrho} \right] = 0, \quad (229)
 \end{aligned}$$

$$\begin{aligned}
 (1 - \sin a \sin \varphi_1) \ddot{\theta} + \frac{1}{2} \sin 2\theta \dot{\varphi}_2^2 + 2 \sin \theta \dot{\varphi}_1 \dot{\varphi}_2 + \sin a \left( -2 \cos 2\varphi_1 \dot{\varphi}_1 \dot{\theta} \right. \\
 + \cos 2\varphi_1 \sin \theta \ddot{\varphi}_2 - 2 \sin 2\varphi_1 \sin \theta \dot{\varphi}_1 \dot{\varphi}_2 - \frac{1}{2} \sin 2\varphi_1 \sin 2\theta \dot{\varphi}_2^2 \Big) \\
 + \cos a \left( \cos 2\varphi_1 \sin \theta \dot{a} \dot{\varphi}_2^2 - \sin 2\varphi_1 \dot{a} \dot{\theta} + \sin \theta \dot{\varphi}_2 \dot{\lambda} \right) \\
 + \frac{2}{\varrho} \left( \dot{\theta} \dot{\varrho} + \sin a \cos 2\varphi_1 \sin \theta \dot{\varphi}_2 \dot{\varrho} - \sin a \sin 2\varphi_1 \dot{\theta} \dot{\varrho} \right) = 0; \quad (230)
 \end{aligned}$$

and finally,

$$\begin{aligned} \dot{\varrho} - \varrho \left[ \frac{1}{4} \dot{a}^2 + \frac{1}{4} \dot{\lambda}^2 + \dot{\varphi}_1^2 + \frac{1}{2} \dot{\theta}^2 + \frac{1}{2} \dot{\varphi}_2^2 + \frac{1}{2} \cos^2 \theta \dot{\varphi}_2^2 + 2 \cos \theta \dot{\varphi}_1 \dot{\varphi}_2 \right. \\ \left. + \sin a \left( \frac{1}{2} \sin 2\varphi_1 \sin^2 \theta \dot{\varphi}_2^2 + \cos 2\varphi_1 \sin \theta \dot{\varphi}_2 \dot{\theta} - \frac{1}{2} \sin 2\varphi_1 \dot{\theta}^2 \right) \right. \\ \left. + \cos a (\dot{\varphi}_1 \dot{\lambda} + \cos \theta \dot{\varphi}_2 \dot{\lambda}) \right] = 0. \end{aligned} \quad (231)$$

A few particular cases are considered below.

1) *Motion of the triangle in the plane:*

$$\dot{\theta} = \dot{\varphi}_2 = 0.$$

In this case the free Lagrangian can be written

$$T = \frac{1}{2} \varrho^2 \left[ \frac{1}{4} \dot{a}^2 + \frac{1}{4} \dot{\lambda}^2 + \cos a \dot{\varphi}_1 \dot{\lambda} + \dot{\varphi}_1^2 \right] + \frac{1}{2} \dot{\varrho}^2, \quad (232)$$

or, remembering that

$$\begin{aligned} p_{\varphi_1} &= \frac{1}{2} \varrho^2 (2\dot{\varphi}_1 + 2 \cos \theta \dot{\varphi}_2 + \cos a \dot{\lambda}), \\ p_{\varphi_2} &= \frac{1}{2} \varrho^2 (\dot{\varphi}_2 + \cos^2 \theta \dot{\varphi}_2 - \sin a \sin 2\varphi_1 \sin^2 \theta \dot{\varphi}_2 \\ &\quad + 2 \cos \theta \dot{\varphi}_1 + \sin a \cos 2\varphi_1 \sin \theta \dot{\theta} + \cos a \cos \theta \dot{\lambda}), \\ p_{\theta} &= \frac{1}{2} \varrho^2 (\dot{\theta} + \sin a \cos 2\varphi_1 \sin \theta \dot{\varphi}_2 - \sin a \sin 2\varphi_1 \dot{\theta}), \end{aligned} \quad (233)$$

in the form

$$T = \frac{1}{\varrho^2} \left[ 2p_a^2 + \frac{1}{\sin^2 a} \left( \frac{1}{2} p_{\varphi_1}^2 - 2p_{\varphi_1} p_{\lambda} \cos a + 2p_{\lambda}^2 \right) \right] + \frac{1}{2} p_{\varrho}^2. \quad (234)$$

The equations of motion are in this case

$$\begin{aligned} \frac{1}{2} \ddot{a} + \frac{1}{\varrho} \dot{\varrho} \dot{a} + \sin a \dot{\varphi}_1 \dot{\lambda} &= 0, \\ \frac{1}{2} \ddot{\lambda} + \frac{1}{\varrho} \dot{\varrho} \dot{\lambda} - \sin a \dot{a} \dot{\varphi}_1 + \cos a \ddot{\varphi} + \frac{2}{\varrho} \cos a \dot{\varphi}_1 \dot{\varrho} &= 0, \\ \ddot{\varphi}_1 + \frac{1}{2} \cos a \dot{\lambda} - \frac{1}{2} \sin a \dot{a} \dot{\lambda} + \frac{1}{\varrho} (2\dot{\varphi}_1 \dot{\varrho} + \cos a \dot{\lambda} \dot{\varrho}) &= 0, \\ \ddot{\varrho} - \varrho \left( \frac{1}{4} \dot{a}^2 + \frac{1}{4} \dot{\lambda}^2 + \dot{\varphi}_1^2 + \cos a \dot{\varphi}_1 \dot{\lambda} \right) &= 0. \end{aligned} \quad (235)$$

2) *Deforming triangle:*

$$\dot{\theta} = \dot{\varphi}_2 = 0, \quad p_{\varphi_1} = 0.$$

The free Lagrangian obtains the form

$$T = \frac{1}{2} \varrho^2 \left( \frac{1}{4} \dot{a}^2 + \frac{1}{4} \dot{\lambda}^2 \sin^2 a \right) + \frac{1}{2} \dot{\varrho}^2 = \frac{2}{\varrho^2} \left( p_a^2 + \frac{1}{\sin^2 a} p_{\lambda}^2 \right) + \frac{1}{2} p_{\varrho}^2. \quad (236)$$

It should be noted that for  $p_\varrho = 0$  this expression has the same form as the Lagrangian of the rotator. We see here an example of the hidden symmetry, which can be generalized to the case of a deforming rotator.

The equations of motion corresponding to the Lagrange function (39) are

$$\begin{aligned}\frac{1}{2}\ddot{a} + \frac{1}{\varrho}\dot{\varrho}\dot{a} - \frac{1}{2}\sin a \cos a \dot{\lambda}^2 &= 0, \\ \sin a \left( \frac{1}{2}\ddot{\lambda} + \frac{1}{\varrho}\dot{\varrho}\dot{\lambda} \right) + \cos a \dot{a}\dot{\lambda} &= 0, \\ \varrho - \varrho \left( \frac{1}{4}\dot{a}^2 + \frac{1}{4}\sin^2 a \dot{\lambda}^2 \right) &= 0.\end{aligned}\tag{237}$$

If we add to the right-hand side forces depending only on  $\varrho$ , we get the equations of a non-rigid rotator.

### 6.3. Potentials for three-body systems

Two examples of interacting particles will be investigated.

#### 6.3.1. The harmonic oscillator potential

The equilibrium state of the three-particle system is an equilateral triangle of side  $\varrho_0$  which can be described by the vectors  $\vec{\xi}_0$  and  $\vec{\eta}_0$ :

$$\begin{aligned}\vec{\xi}_0 &= \begin{pmatrix} 0 \\ \frac{\varrho_0}{\sqrt{2}} \end{pmatrix}, & \vec{\eta} &= \begin{pmatrix} \frac{\varrho_0}{\sqrt{2}} \\ 0 \end{pmatrix}, \\ \vec{\xi}_0 \vec{\eta}_0 &= 0, & \xi_0^2 - \eta_0^2 &= 0.\end{aligned}\tag{238}$$

The parameters  $a_0$  and  $\lambda_0$  corresponding to the equilibrium state will have the values

$$a_0 = \pi, \quad \lambda_0 = 0,$$

and consequently

$$\vec{z}_0 = \frac{\varrho_0}{\sqrt{2}} \left( e^{i(\pi/2)} \vec{l}_1 + i e^{-i(\pi/2)} \vec{l}_2 \right).\tag{239}$$

Consider the motion of the three particles with the potential energy

$$U = \frac{1}{2} [(\xi - \xi_0)^2 + (\eta - \eta_0)^2] = \frac{1}{2} \left( \varrho^2 + \varrho_0^2 - 2\varrho\varrho_0 \sin \frac{a}{2} \cos \lambda \right).\tag{240}$$

From the expression

$$F_t = -\frac{\partial U}{\partial q_i},\tag{241}$$

we obtain

$$\begin{aligned} F_a &= \frac{1}{2} \varrho \varrho_0 \cos \frac{a}{2} \cos \frac{\lambda}{2}, \\ F_\lambda &= -\frac{1}{2} \varrho \varrho_0 \sin \frac{a}{2} \sin \frac{\lambda}{2}, \\ F_\varrho &= -\varrho + \varrho_0 \sin \frac{a}{2} \cos \frac{\lambda}{2}, \\ F_{\varphi_1} &= F_{\varphi_2} = F_\theta = 0. \end{aligned} \quad (242)$$

Constructing  $L = T - U$ , it is now easy to get the equations of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q_l} = 0. \quad (243)$$

The equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_l} - \frac{\partial L}{\partial \varphi_l} = 0 \quad (244)$$

(where  $\varphi_3 = \theta$ ) stay unchanged; instead of Eqs. (226), (227) and (231) we obtain

$$\begin{aligned} \frac{1}{2} \ddot{a} - \cos a \left( \frac{1}{2} \sin 2\varphi_1 \sin^2 \theta \dot{\varphi}_2^2 + \cos 2\varphi_1 \sin \theta \dot{\varphi}_2 \dot{\theta} - \frac{1}{2} \sin 2\varphi_1 \dot{\theta}^2 \right) \\ + \sin a (\dot{\varphi}_1 \dot{\lambda} + \cos \theta \dot{\varphi}_2 \dot{\lambda}) + \frac{1}{\varrho} \dot{\varrho} \dot{a} - \frac{\varrho_0}{\varrho} \cos \frac{a}{2} \cos \frac{\lambda}{2} = 0, \end{aligned} \quad (245)$$

$$\begin{aligned} \frac{1}{2} \ddot{\lambda} - \sin a (\dot{a} \dot{\varphi}_1 + \cos \theta \dot{a} \dot{\varphi}_2) + \cos a (\varphi_1 - \sin \theta \dot{\theta} \dot{\varphi}_2 + \cos \theta \dot{\varphi}_2) + \\ + \frac{1}{\varrho} \dot{\varrho} \dot{\lambda} + \frac{2}{\varrho} \cos a (\dot{\varphi}_1 \dot{\varrho} + \cos \theta \dot{\varphi}_2 \dot{\varrho}) + \frac{\varrho_0}{\varrho} \sin \frac{a}{2} \sin \frac{\lambda}{2} = 0, \end{aligned} \quad (246)$$

and

$$\begin{aligned} \varrho - \varrho \left[ \frac{1}{4} \dot{a}^2 + \frac{1}{4} \dot{\lambda}^2 + \dot{\varphi}_1^2 + \frac{1}{2} \dot{\theta}^2 + \frac{1}{2} \dot{\varphi}_2^2 + \frac{1}{2} \cos^2 \theta \dot{\varphi}_2^2 + 2 \cos \theta \dot{\varphi}_1 \dot{\varphi}_2 \right. \\ + \sin a \left( \frac{1}{2} \sin 2\varphi_1 \sin^2 \theta \dot{\varphi}_2^2 + \cos 2\varphi_1 \sin \theta \dot{\varphi}_2 \dot{\theta} - \frac{1}{2} \sin 2\varphi_1 \dot{\theta}^2 \right) \\ \left. + \cos a (\dot{\varphi}_1 \dot{\lambda} + \cos \theta \dot{\varphi}_2 \dot{\lambda}) \right] + \varrho - \varrho_0 \sin \frac{a}{2} \cos \frac{\lambda}{2} = 0. \end{aligned} \quad (247)$$

Let us consider again the case of a non-rotating triangle:

$$\dot{\theta} = \dot{\varphi}_2 = 0 \quad \text{and} \quad p_{\varphi_1} = 0, \quad \text{i.e.} \quad \dot{\varphi}_1 = -\frac{1}{2} \cos a \dot{\lambda}. \quad (248)$$

The equations of motion assume the form

$$\begin{aligned} \frac{1}{2} \dot{a} + \frac{1}{\varrho} \dot{\varrho} \dot{a} - \frac{1}{2} \sin a \cos a \dot{\lambda}^2 \frac{\varrho_0}{\varrho} \cos \frac{a}{2} \frac{\lambda}{2} = 0, \\ \sin^2 a \left( \frac{1}{2} \dot{\lambda} + \frac{1}{\varrho} \dot{\varrho} \dot{\lambda} \right) + \sin a \cos a \dot{a} \dot{\lambda} + \frac{\varrho_0}{\varrho} \sin \frac{a}{2} \sin \frac{\lambda}{2} = 0, \\ \dot{\varrho} - \varrho \left( \frac{1}{4} \dot{a}^2 + \frac{1}{4} \sin^2 a \dot{\lambda}^2 \right) + \varrho - \varrho_0 \sin \frac{a}{2} \cos \frac{\lambda}{2} = 0. \end{aligned} \quad (249)$$

If in this case we take  $\varrho_0 = 0$ , we will have

$$\begin{aligned} \frac{1}{2}\ddot{a} + \frac{1}{\varrho}\dot{\varrho}\dot{a} - \frac{1}{2}\sin a \cos a \dot{\lambda}^2 &= 0, \\ \sin a \left( \frac{1}{2}\dot{\lambda} + \frac{1}{\varrho}\dot{\varrho}\dot{\lambda} \right) + \cos a \dot{a}\dot{\lambda} &= 0, \\ \ddot{\varrho} - \varrho \left( \frac{1}{4}\dot{a}^2 + \frac{1}{4}\sin^2 a \dot{\lambda}^2 \right) + \varrho &= 0. \end{aligned} \quad (250)$$

As an example, let us consider solutions for constant  $\lambda$  and  $a$ , illustrating a few cases of the deformed triangle in detail. The projections of the radius vectors  $\vec{x}_l$  onto the axes  $\vec{l}_1$  and  $\vec{l}_2$  are as follows:

$$\begin{aligned} x_1^{(1)} &= \frac{\varrho}{2} \left( \sin \frac{a-\lambda}{2} - \frac{1}{\sqrt{3}} \cos \frac{a-\lambda}{2} \right), \\ x_2^{(1)} &= -\frac{\varrho}{2} \left( \sin \frac{a-\lambda}{2} + \frac{1}{\sqrt{3}} \cos \frac{a-\lambda}{2} \right), \\ x_3^{(1)} &= \frac{\varrho}{\sqrt{3}} \cos \frac{a-\lambda}{2}, \end{aligned} \quad (251)$$

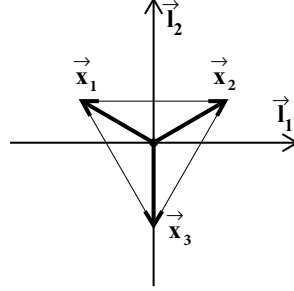
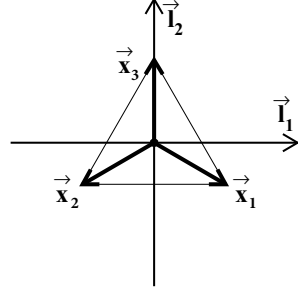
and

$$\begin{aligned} x_1^{(2)} &= \frac{\varrho}{2} \left( \cos \frac{a+\lambda}{2} - \frac{1}{\sqrt{3}} \sin \frac{a+\lambda}{2} \right), \\ x_2^{(2)} &= -\frac{\varrho}{2} \left( \cos \frac{a+\lambda}{2} + \frac{1}{\sqrt{3}} \sin \frac{a+\lambda}{2} \right), \\ x_3^{(2)} &= \frac{\varrho}{\sqrt{3}} \sin \frac{a+\lambda}{2}. \end{aligned} \quad (252)$$

If  $a = \text{const}$ ,  $\lambda = \text{const}$  and  $\varrho_0 \neq 0$ , we obtain from (249)

$$\begin{aligned} \frac{\varrho_0}{\varrho} \cos \frac{a}{2} \cos \frac{\lambda}{2} &= 0, \\ \frac{\varrho_0}{\varrho} \sin \frac{a}{2} \sin \frac{\lambda}{2} &= 0, \\ \varrho + \varrho - \varrho_0 \sin \frac{a}{2} \cos \frac{\lambda}{2} &= 0. \end{aligned} \quad (253)$$

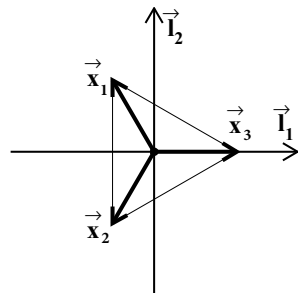
It can be easily seen that in this case  $a$  and  $\lambda$  are multiples of  $\pi$ , and only similarity transformations are possible; for example:



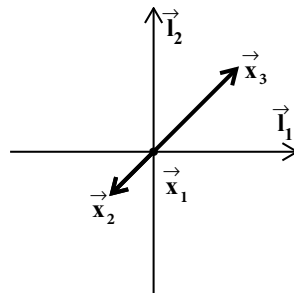
$$\begin{aligned}
 \lambda = 0, \quad a = \pi, \\
 x_1^{(1)} &= \frac{\varrho}{2}, \\
 x_2^{(1)} &= \frac{\varrho}{2}, \\
 x_3^{(1)} &= 0 \\
 x_1^{(2)} &= \frac{1}{2\sqrt{3}} \varrho, \\
 x_2^{(2)} &= -\frac{1}{2\sqrt{3}} \varrho, \\
 x_3^{(2)} &= \frac{\varrho}{\sqrt{3}}, \\
 \lambda = 0, \quad a = 3\pi \\
 x_1^{(1)} &= \frac{\varrho}{2}, \\
 x_2^{(1)} &= \frac{\varrho}{2}, \\
 x_3^{(1)} &= 0, \\
 x_1^{(2)} &= \frac{\varrho}{2\sqrt{3}}, \\
 x_2^{(2)} &= \frac{\varrho}{2\sqrt{3}}, \\
 x_3^{(2)} &= \frac{\varrho}{\sqrt{3}}.
 \end{aligned} \tag{254}$$

If, on the contrary,  $\varrho = 0$ , then the fixed value of  $a$  still does not determine the value of  $\lambda$ , so that arbitrary deformations are possible. We give in the following a few examples:

$$\begin{aligned}
 \text{a) } \lambda = a \quad x_1^{(1)} &= -\frac{\varrho}{2\sqrt{3}}, & x_1^{(2)} &= \frac{\varrho}{2} \left( \cos a - \frac{1}{\sqrt{3}} \sin a \right), \\
 x_2^{(1)} &= -\frac{\varrho}{2\sqrt{3}}, & x_2^{(2)} &= -\frac{\varrho}{2} \left( \cos a + \frac{1}{\sqrt{3}} \sin a \right), \\
 x_3^{(1)} &= \frac{\varrho}{\sqrt{3}}, & x_3^{(2)} &= \frac{\varrho}{\sqrt{3}} \sin a.
 \end{aligned}$$

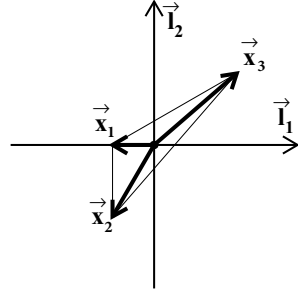


$$\lambda = a = 0,$$



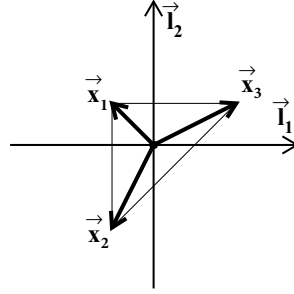
$$\lambda = a = \frac{\pi}{2},$$

$$x_1^{(2)} = \frac{\varrho}{2}, \quad x_2^{(2)} = \frac{\varrho}{2}, \quad x_3^{(2)} = 0, \quad x_1^{(2)} = -\frac{\varrho}{2\sqrt{3}}, \quad x_2^{(2)} = -\frac{\varrho}{2\sqrt{3}}, \quad x_3^{(2)} = \frac{\varrho}{2}.$$



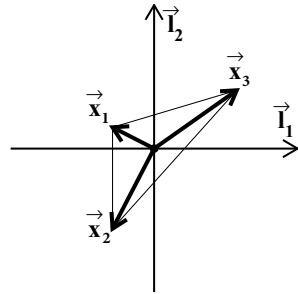
$$\lambda = a = \frac{\pi}{3},$$

$$x_1^{(2)} = 0, \quad x_2^{(2)} = \frac{\varrho}{2}, \quad x_3^{(2)} = \frac{\varrho}{2},$$



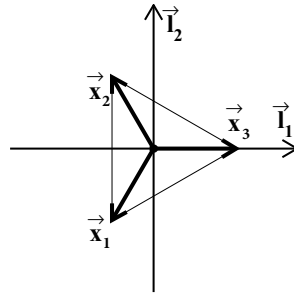
$$\lambda = a = \frac{\pi}{6},$$

$$x_1^{(2)} = \frac{\varrho}{2\sqrt{3}}, \quad x_2^{(2)} = \frac{\varrho}{\sqrt{3}}, \quad x_3^{(2)} = \frac{\varrho}{2\sqrt{3}}.$$



$$\lambda = a = \frac{\pi}{4},$$

$$\begin{aligned} x_1^{(2)} &= \frac{\varrho}{2\sqrt{3}} \left( 1 - \frac{1}{2\sqrt{3}} \right), \\ x_2^{(2)} &= -\frac{\varrho}{2\sqrt{2}} \left( 1 + \frac{1}{\sqrt{3}} \right), \\ x_3^{(2)} &= \frac{\varrho}{\sqrt{6}}, \end{aligned}$$

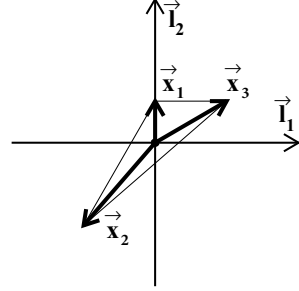


$$\lambda = a = \pi,$$

$$\begin{aligned} x_1^{(2)} &= -\frac{\varrho}{2}, \\ x_2^{(2)} &= \frac{\varrho}{2}, \\ x_3^{(2)} &= 0. \end{aligned}$$

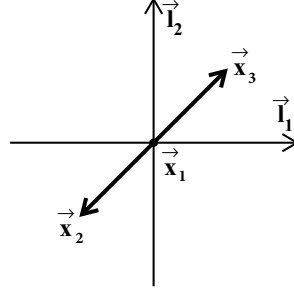
**b)**  $a - \lambda = \frac{\pi}{3}, \quad x_1^{(3)} = 0, \quad x_1^{(2)} = \frac{\varrho}{2} \left( \cos \left( \lambda + \frac{\pi}{6} \right) \frac{1}{\sqrt{3}} \sin \left( \lambda + \frac{\pi}{6} \right) \right),$   
 $x_2^{(1)} = -\frac{\varrho}{2}, \quad x_2^{(2)} = \frac{\varrho}{2} \left( \cos \left( \lambda + \frac{\pi}{6} \right) + \frac{1}{\sqrt{3}} \sin \left( \lambda + \frac{\pi}{6} \right) \right),$   
 $x_3^{(1)} = \frac{\varrho}{2}, \quad x_3^{(2)} = \frac{\varrho}{\sqrt{3}} \sin \left( \lambda + \frac{\pi}{6} \right),$





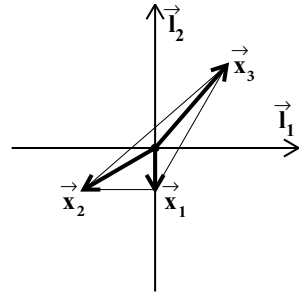
$$\lambda = 0, \quad a = \frac{\pi}{3},$$

$$x_1^{(2)} = \frac{\rho}{2\sqrt{3}}, \quad x_2^{(2)} = -\frac{\rho}{\sqrt{3}}, \quad x_3^{(2)} = \frac{\rho}{2\sqrt{3}},$$



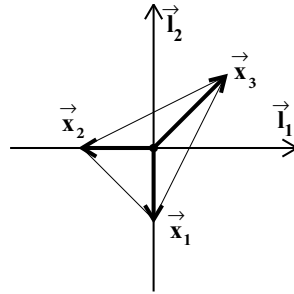
$$\lambda = \frac{\pi}{6}, \quad a = \frac{\pi}{2},$$

$$x_1^{(2)} = 0, \quad x_2^{(2)} = \frac{\rho}{2}, \quad x_3^{(2)} = \frac{\rho}{2}.$$



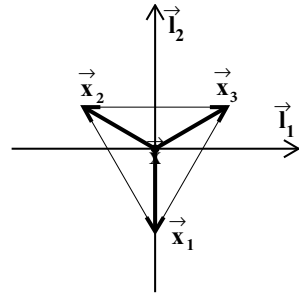
$$\lambda = \frac{\pi}{3}, \quad a = \frac{2\pi}{3},$$

$$x_1^{(2)} = -\frac{\rho}{2\sqrt{3}}, \quad x_2^{(2)} = \frac{\rho}{2\sqrt{3}}, \quad x_3^{(2)} = \frac{\rho}{\sqrt{3}},$$



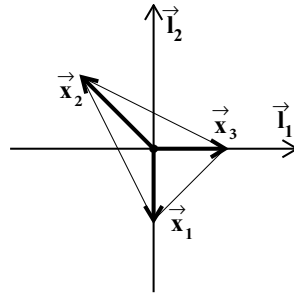
$$\lambda = \frac{\pi}{3}, \quad a = \frac{5\pi}{6},$$

$$x_1^{(2)} = -\frac{\rho}{2}, \quad x_2^{(2)} = 0, \quad x_3^{(2)} = \frac{\rho}{2}.$$



$$\lambda = \frac{2\pi}{3}, \quad a = \pi,$$

$$x_1^{(2)} = -\frac{\rho}{\sqrt{3}}, \quad x_2^{(2)} = \frac{\rho}{2\sqrt{3}}, \quad x_3^{(2)} = \frac{\rho}{2\sqrt{3}},$$



$$\lambda = \frac{5\pi}{6}, \quad a = \frac{7\pi}{6},$$

$$x_1^{(2)} = -\frac{\rho}{2\sqrt{3}}, \quad x_2^{(2)} = -\frac{\rho}{2\sqrt{3}}, \quad x_3^{(2)} = \frac{\rho}{\sqrt{3}}.$$

### 6.3.2. Three-body problem in celestial mechanics (the Laplace case).

#### Self-consistent field in classical mechanics

Suppose that an attractive Newtonian potential is acting between three particles. It can be shown that there exists a solution for which all three particles stay in

the vertices of an equilateral triangle while each particle moves along an elliptic trajectory about the common center-of-mass as if there was a central body the mass of which is equal to the sum of masses of the three particles.

Assume that  $\dot{a}\dot{\theta} = \dot{\varphi}_2$  is equal to zero. If the particles form an equilateral triangle, we have  $a = 0$ , and the distance between the particles is  $\varrho$ . The potential energy in this case is equal to  $U = -3/\varrho$ , so that the equations of motion take the form

$$\varrho - \varrho \left[ \frac{1}{4} \dot{\lambda}^2 + \dot{\varphi}_1^2 + \dot{\varphi}_1 \dot{\lambda} \right] = 0, \quad \frac{1}{2} \ddot{\lambda} + \dot{\varphi}_1 + \frac{\dot{\varrho}}{\varrho} \dot{\lambda} + \frac{2}{\varrho} \dot{\varphi}_1 \dot{\varrho} = 0. \quad (255)$$

Introducing a new variable

$$\dot{\psi} = \frac{1}{2} \dot{\lambda} + \dot{\varphi}_1, \quad (256)$$

we obtain the Kepler equations

$$\dot{\varrho} \varrho \dot{\psi}^2 + \frac{3}{\varrho^2} = 0, \quad \psi + \frac{2}{\varrho} \dot{\varrho} \dot{\psi} = 0. \quad (257)$$

If we now express  $\vec{x}_1, \vec{x}_2$  and  $\vec{x}_3$  in terms of  $\varrho$ , we get the equations of three ellipses. It is easy to prove that in the case  $a \neq 0$  the equations will not lead to the Kepler equations. It should be noted that the type of the solution is independent of the form of the potential; indeed, we use only the fact that the forces acting on any of the three particles are directed to the centre-of-mass of the triangle.

## 7. Conclusion

The present review paper is, in a way, an addition to the book,<sup>29</sup> considering the three-body problem from a different point of view: we apply here the group theoretical method to its investigation.

The Hamiltonian of the three-body problem consists of a kinematic part and a part due to the interactions. Here we investigate mainly the kinematic part. The group theoretical properties of low-lying states are defined by the structure of the interaction while in the higher excitations the kinematic part of the components is prevailing. The investigation of the highly excited states is rather relevant in different actual problems. As an example we may consider the three-quark baryon states: the quark model predicts, obviously, more bound states than can be observed experimentally. Another interesting problem is that of the molecular systems with short range pair-interactions which lead to three-particle systems with binding energies close to zero.<sup>14</sup> In such loose systems the details of the short range interactions are, as a rule, not too important, while the quantum numbers of pair interactions may turn out to be essential.

In the present paper we have described the quantum mechanical consideration of the kinematical part of the Hamiltonian. The quantum mechanical investigation can be easily generalized to the relativistic case. The reason is that in the general case the kinematical parts have the same structure, namely:  $s_{12} + s_{13} + s_{23} \longrightarrow \vec{k}_{12}^2 + \vec{k}_{13}^2 + \vec{k}_{23}^2$  (let us remind here that the masses can be considered to be equal).

The non-relativistic operator can be easily changed to a relativistic one, *i.e.* written in a covariant form. The covariant form of the operators, similar to the operators of angular variables, were considered in detail for two-particle systems see Ref. 29.

There are no principal difficulties in writing three-particle angular momentum operators. In other words, the wave functions of the composite systems investigated here can be successfully applied to relativistic problems.

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